

A fast convergence theorem for nearly multiplicative connections on proper Lie groupoids

Giorgio Trentinaglia

Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Abstract

Motivated by the study of global geometric properties of differentiable stacks presented by proper Lie groupoids, we investigate the existence of multiplicative connections on such groupoids. We show that one can always deform a given connection which is only approximately multiplicative into a genuinely multiplicative connection. The proof of this fact presented here relies on a recursive averaging technique. We regard our results as a preliminary step towards the construction of an obstruction theory for multiplicative connections on proper Lie groupoids.

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Introduction

This paper is supposed to be the first of a series devoted to the study of multiplicative connections on proper Lie groupoids. These connections provide a key to the understanding of the “global transversal geometry” of proper Lie groupoids (that is, roughly speaking, the global geometry of their “orbit spaces” viewed as “generalized reduced

MSC (2010): Primary 58H05; Secondary 22A22; 53C05; 58C30.

Keywords: proper Lie groupoid; multiplicative connection; pseudo-representation; C^k -topology; normalized Haar system.

Part of the results contained in the present article were obtained while the author was a guest of the Max Planck Institute for Mathematics in Bonn, Germany. The author acknowledges support from the Portuguese Foundation for Science and Technology (Fundação para a Ciência e a Tecnologia) through Postdoctoral Grant # SFRH/BPD/81810/2011 and, partly, through Grant # UID/MAT/04459/2013.

orbifolds” [27]) in that they make it possible to construct *effective representations* [26] of such groupoids. The basic link between connections and representations is provided by the following remark: the *effect* of a multiplicative connection, that is, the representation obtained by composing the horizontal lift of the connection with the differential of the target mapping of the groupoid, is an effective representation. Whenever effective representations can be found, one can generalize the classical result in the theory of reduced orbifolds that any effective, proper, étale, Lie groupoid is equivalent (in the sense of Morita) to the translation groupoid associated to some compact Lie group action [22, 14]; this result and its variants [13] are generally known as *presentation theorems*. The reader is referred to the introduction of [27] for a more elaborate exposition of these ideas. Results concerning the existence of effective representations of proper Lie groupoids usually entail criteria under which the same groupoids admit *faithful* representations. Such criteria are especially relevant to algebraic topology since faithful representations enable one to build simpler models of cohomology theories such as equivariant K -theory [17, 8].

Multiplicative connections are interesting for many other reasons. To begin with, they are required for various constructions, e.g. in the theory of differentiable stacks or in quantization [2, 25, 12]. Secondly, they enable one to make sense of the adjoint representation of a Lie groupoid as an ordinary graded representation, rather than just as a representation up to homotopy [1], and thus to introduce relevant simplifications in the computation of the cohomology of the classifying space of a Lie groupoid or in the calculation of certain cohomological obstructions such as intrinsic secondary characteristic classes [9, 3, 4]. Thirdly, they convey non-trivial information about the structure of the Lie groupoids on which they are defined; see [5, Corollary 3.12] for instance. Because of these many applications, we want to understand how one may actually *construct* multiplicative connections, and what the topological properties of the space formed by all such connections are. The present work develops the analytic foundations of an obstruction theory for multiplicative connections on proper Lie groupoids which is supposed to be the subject of future investigations.

Let us begin by giving a quick description of the main results of the paper. We say that an Ehresmann connection on the source fibration of a Lie groupoid is *non-degenerate*, if the associated pseudo-representation of the groupoid on the tangent bundle of its own base manifold (that is, the pseudo-representation obtained by composing the horizontal lift of the connection with the differential of the target mapping of the groupoid) is invertible. We say that it is *effective*, if the same pseudo-representation is a representation viz. respects units and composition. We point out that any proper Lie groupoid admits an *averaging operator* which is defined on non-degenerate connections and has the property that the average of an effective connection is a multiplicative connection (a connection which, as a distribution, constitutes a subgroupoid of the tangent groupoid of the given groupoid). This is the content of Proposition 3.7 below. If a connection is already multiplicative, it is left unchanged by the averaging operator. When one is only given some non-degenerate connection which is not effective, one may in principle still consider the associated sequence of averaging iterates obtained by recursive application of the averaging operator. Provided the initial connection is, in a suitable technical sense that we make precise with Definition 5.1, “close enough” to being effective—in which case we say, slightly incorrectly, that it is *nearly multiplicative*—it turns out that

the associated sequence of averaging iterates is (actually defined and) convergent to a (unique) multiplicative connection. This is the fast convergence theorem mentioned in the title, which is stated below as Theorem 6.1. Our fast convergence theorem for nearly multiplicative connections is essentially a corollary to a similar fast convergence theorem for nearly multiplicative pseudo-representations, Theorem 5.2.

As we have already commented, the present study is part of a general plan aimed at understanding the precise obstructions to the existence of multiplicative connections on proper Lie groupoids. Our hope is that these obstructions may eventually be used to manufacture new vector bundles (other than tangent bundles of groupoid bases) on which proper Lie groupoids may act effectively. A strategy which we regard as promising in this respect is to work stratum after stratum starting from the “maximally singular locus” of a groupoid (the set of all base points that lie on orbits of minimal dimension). As a foretaste of what role the fast convergence theorem is going to play in our plans, consider an arbitrary source-proper Lie groupoid. If we fix a connection at random, there will be an invariant open neighborhood of the “semi-stable locus” of the groupoid (the set of all base points that lie on zero-dimensional orbits) such that the connection is nearly multiplicative over this open neighborhood. Our theorem then implies that over the same neighborhood there must be a multiplicative connection. With some extra work, one can show that for a general proper Lie groupoid there is always some invariant open neighborhood of the maximally singular locus with the property that the groupoid admits an effective representation over this neighborhood (on some vector bundle which in general need not be a tangent bundle).

There are other applications of our theorem which are worth mentioning. Let Φ_0 and Φ_1 be multiplicative connections on a proper Lie groupoid which has only zero-dimensional orbits. Note that on any such groupoid every connection is automatically effective. Since groupoid connections form an affine Fréchet manifold, Φ_0 and Φ_1 can be joined by means of a line segment of groupoid connections $\Phi_t = (1 - t)\Phi_0 + t\Phi_1$. We can apply our averaging operator to each connection Φ_t and thus obtain a smooth path of multiplicative connections which deforms Φ_1 into Φ_0 . The space of multiplicative connections on any proper Lie groupoid with only discrete orbits is therefore path-connected and, in fact, contractible. More interestingly consider an arbitrary multiplicative connection Φ_0 on a compact Lie groupoid. If Φ_1 is another multiplicative connection which is “sufficiently close” to Φ_0 , then all the connections Φ_t on the line segment joining Φ_1 and Φ_0 will be nearly multiplicative. Therefore, by our fast convergence theorem, there will be a continuous path of multiplicative connections deforming Φ_1 into Φ_0 . The space of multiplicative connections on a compact Lie groupoid is thus semi-locally path-connected and, in fact, semi-locally contractible. These remarks illustrate the usefulness of our fast convergence theorem in the study of the topological properties of the space of multiplicative connections.

To conclude, a few words about the logical organization of our paper. The first two sections are essentially introductory, and contain little, if any, original material. They serve the purpose of fixing the terminology and of recalling the basic notions that are used throughout the rest of the paper. Section 2 includes a discussion of multiplicative connections through examples which are designed to help the reader’s intuition. In principle the two appendices can be read immediately after Section 1, as they do not depend on the rest of the paper. In Appendix A we recall some basic facts about the C^k -

topology on the space of cross-sections of a vector bundle, needed for the proofs of our main theorems. Appendix B provides a self-contained introduction to Haar integration. It includes some results, notably Proposition B.12 and B.13, which, in the form required for our purposes, are not so easy to find in the literature; the former proposition already plays a role in Section 3, whereas the latter only starting from Section 5. In Section 3 we introduce the main original construction of our paper namely the averaging operator for groupoid connections (Definition 3.3). In Section 4 we introduce a similar averaging operator for pseudo-representations (30) and study its uniform convergence properties under recursive application. The last two sections, 5 and 6, are entirely devoted to the proofs of our two main theorems (Theorem 5.2 and 6.1).

1. Groupoid connections and pseudo-representations

By a *differentiable manifold* we shall mean a (non-empty) locally compact C^∞ -manifold. (We assume that the reader is familiar with the rudiments of differential geometry at the level, say, of the first few chapters of Lang's "*Fundamentals*" [16].) A general differentiable manifold may not be Hausdorff nor second countable, and may have components modeled on different (finite-dimensional real) vector spaces. We shall say that a differentiable manifold is *smooth* whenever it is Hausdorff, second countable, and of constant dimension. A *differentiable mapping* will be a mapping of class C^∞ between differentiable manifolds, and a *smooth mapping* will be one between smooth manifolds.

By a *differentiable groupoid* we shall mean a small groupoid $\Gamma \rightrightarrows X$ in which Γ and X are differentiable manifolds, the source s^Γ and the target t^Γ are submersive differentiable mappings, and the other structure mappings (namely the composition m^Γ , the unit u^Γ and the inverse i^Γ) are differentiable. We shall call *Lie groupoid* any differentiable groupoid $\Gamma \rightrightarrows M$ in which M is *smooth* and in which Γ is *second countable*. A *homomorphism* of differentiable groupoids will be a differentiable functor. A differentiable groupoid $\Gamma \rightrightarrows X$ will be said to be *Hausdorff*, resp., *second countable*, resp., *of constant dimension* whenever the corresponding property holds for both Γ and X , and *essentially connected* if the only non-empty, open, closed, Γ -invariant subset $U \subset X$ is $U = X$ itself.

The notion of Lie groupoid we adopt here differs from that given in the standard textbook [18] in two respects. First, we require second countability. Second, we allow the arrow manifold to have components of different dimensions. The latter convention is a very natural one in the context of representation theory, for reasons which we do not intend to discuss here. As to second countability, we point out that in most classical textbooks Lie groups are supposed to be second countable. In [18] the authors also postulate that the source fibers of a Lie groupoid ought to be Hausdorff; this hypothesis is redundant (compare below).

Fundamental structure theorem. *Let $\Gamma \rightrightarrows X$ be an arbitrary differentiable groupoid. The following statements hold:*

- (a) *For each pair of base points $x, y \in X$ the subset $\Gamma_y^x = \Gamma(x, y) \subset \Gamma$ is a differentiable submanifold.*

- (b) For each base point $x \in X$ the isotropy group $\Gamma_x^x = \Gamma(x, x)$, with the differentiable structure inherited from Γ according to (a), is a differentiable group.
- (c) For each base point $x \in X$ the composition of arrows restricts to a free, differentiable, right action $\Gamma^x \times \Gamma_x^x \rightarrow \Gamma^x$ of the isotropy group Γ_x^x on the source fiber $\Gamma^x = \Gamma(x, -) \subset \Gamma$. This action has the property that there exists a (unique) differentiable manifold structure on the quotient set Γ^x / Γ_x^x which makes the quotient projection $\Gamma^x \rightarrow \Gamma^x / \Gamma_x^x$ into a submersion.

Proof. The first statement can be proved as in [18]. The last statement is essentially a consequence of Godement's theorem, a proof of which (valid for arbitrary differentiable manifolds) can be found in [24]. The details are left to the reader. \square

Let $\Gamma \rightrightarrows X$ be an arbitrary differentiable groupoid. For each base point $x \in X$ the source fiber Γ^x is a differentiable submanifold of Γ . The differentiable group $G_x = \Gamma_x^x$ acts differentiably and freely from the right on Γ^x , and there is a unique differentiable structure on the quotient Γ^x / G_x which makes the projection $pr_x^\Gamma : \Gamma^x \rightarrow \Gamma^x / G_x$ into a submersion. We shall indicate the resulting differentiable manifold by O_x^Γ . This manifold is injectively immersed into X in a canonical fashion, namely, there is a unique map in_x^Γ of O_x^Γ into X such that $in_x^\Gamma \circ pr_x^\Gamma = t^\Gamma | \Gamma^x$, and this map is necessarily differentiable, injective, and immersive. We shall refer to $in_x^\Gamma : O_x^\Gamma \hookrightarrow X$ as the *orbit* of Γ (shortly, Γ -*orbit*) through x . We shall also refer to the set-theoretic image $\text{im}(in_x^\Gamma) \subset X$ by means of the notation Γx . The differentiable right G_x -bundle $pr_x^\Gamma : \Gamma^x \twoheadrightarrow O_x^\Gamma$ is equivariantly locally trivial. Indeed, any local differentiable section γ to pr_x^Γ induces an equivariant local trivialization of this bundle according to the prescription $(a, g) \mapsto \gamma(a)g$, and any equivariant local trivialization is of this form for a unique local differentiable section. If the base manifold X is Hausdorff then by the local triviality of pr_x^Γ so must be the source fiber Γ^x (for in that case the orbit O_x^Γ must be Hausdorff, because it is injectively immersed into X , whereas G_x , being a differentiable group, is always Hausdorff). One can use local triviality, moreover, to show that if X is of constant dimension then the same must be true of O_x^Γ and Γ^x . Thus, in particular, any essentially connected Lie groupoid must be of constant dimension.

Let \mathbb{K} denote a fixed number field, which we shall always suppose to be either the real numbers \mathbb{R} or the complex numbers \mathbb{C} . By a (\mathbb{K} -linear) *differentiable vector bundle* over a differentiable manifold X we shall mean a triplet $E = (p_E, +_E, \cdot_E)$ consisting of the following data: (i) a surjective submersion between differentiable manifolds $p_E : E \twoheadrightarrow X$ (the abuse of notation is intentional here); (ii) a differentiable mapping $+_E : E \times_X E \rightarrow E$ such that $p_E(e +_E e') = p_E(e) = p_E(e')$ for all $(e, e') \in E \times_X E$, called 'sum', and a differentiable mapping $\cdot_E : \mathbb{K} \times E \rightarrow E$ such that $p_E(a \cdot_E e) = p_E(e)$ for all $(a, e) \in \mathbb{K} \times E$, called 'multiplication by scalars', which make each fiber $E_x := p_E^{-1}(x)$ into a vector space over \mathbb{K} . When $\mathbb{K} = \mathbb{R}$ (resp., $\mathbb{K} = \mathbb{C}$), we shall also refer to E as a *real* (resp., *complex*) differentiable vector bundle. When no danger of confusion is likely to arise, we shall omit the subscript ' E ' from ' p_E ', ' $+_E$ ' or ' \cdot_E ' and simply write ' ae ' instead of ' $a \cdot_E e$ '. A *morphism* $\alpha : E \rightarrow F$ between two \mathbb{K} -linear differentiable vector bundles E and F over a given differentiable manifold X will be a differentiable mapping α of E into F which satisfies the condition $p_F \circ \alpha = p_E$ and which for each point $x \in X$ induces a \mathbb{K} -linear map α_x of E_x into F_x . With this notion of morphism, the

\mathbb{K} -linear differentiable vector bundles over X form a category. One can show that every \mathbb{K} -linear differentiable vector bundle E over X is locally trivial, of locally finite rank; that is to say, for each point $x \in X$ there exist an open neighborhood $U \ni x$ in X and an isomorphism of \mathbb{K} -linear differentiable vector bundles over U

$$E|U := p_E^{-1}(U) \xrightarrow{\sim} U \times \mathbb{K}^r.$$

We shall not digress into a proof of this fact here; the reader may regard the property of local triviality as being part of the definition, if they prefer to do so. The non-negative integer $r = \dim_{\mathbb{K}} E_x$ will be called the *rank* of E at x and denoted by $\text{rk}_E(x)$. When the locally constant function $\text{rk}_E : X \rightarrow \mathbb{N}$ is overall constant, we shall say that E is of *constant rank*. Local triviality implies that if E is a differentiable vector bundle over a *smooth* manifold, then E is a vector bundle of constant rank if and only if its total space is a smooth manifold. We shall call *smooth vector bundle* any differentiable vector bundle (over a smooth manifold) which satisfies these two equivalent conditions.

Let X be a differentiable manifold and let E be a \mathbb{K} -linear differentiable vector bundle over X . For each order of differentiability $k = 0, 1, 2, \dots, \infty$ and for each differentiable submanifold Y of X , the notation $\Gamma^k(Y; E)$ will be used to indicate the vector space (over \mathbb{K}) formed by all cross-sections of class C^k of E over Y , that is to say, all maps $\xi : Y \rightarrow E$ of class C^k such that $p_E \circ \xi = \text{in}_Y^X$, where $\text{in}_Y^X : Y \hookrightarrow X$ denotes the inclusion of Y into X . When Z is another submanifold of X which is contained in Y , we shall let res_Z^Y denote the linear map of $\Gamma^k(Y; E)$ into $\Gamma^k(Z; E)$ given by $\xi \mapsto \xi|Z$ (restriction over Z).

By a *connection* on a differentiable groupoid $\Gamma \rightrightarrows X$ we shall mean an (Ehresmann) connection on the groupoid source fibration $s = s^F : \Gamma \twoheadrightarrow X$. Explicitly, a connection on $\Gamma \rightrightarrows X$ is a right splitting η for the following short exact sequence of morphisms of differentiable vector bundles over the manifold Γ :

$$0 \longrightarrow T^\uparrow \Gamma \xrightarrow{\subseteq} T\Gamma \xrightarrow[\eta]{s_*} s^*TX \longrightarrow 0$$

(here $T^\uparrow \Gamma$ denotes the s -vertical subbundle of the tangent bundle of the manifold Γ , defined by setting $T_g^\uparrow \Gamma := \ker T_g s = T_g \Gamma^{sg}$, and s_* denotes the unique morphism that corresponds to the tangent source mapping $Ts : T\Gamma \rightarrow TX$ by virtue of the universal property of the vector bundle pullback s^*TX ; ‘right splitting’ means ‘ $s_* \circ \eta = \text{id}_{s^*TX}$ ’). We shall usually identify η with the subbundle $H = \text{im } \eta$ of the tangent bundle $T\Gamma$ (here ‘ H ’ is supposed to be read as ‘capital η ’) and refer to $\eta = \eta^H := (s_*|H)^{-1}$ as the *horizontal lift* induced by H . We shall also refer to $\beta^H := \text{id}_{T\Gamma} - \eta^H \circ s_* : T\Gamma \rightarrow T^\uparrow \Gamma$ as the *vertical projection* induced by H . If the condition $\eta_{1x} = T_x 1$ ($T_x X \rightarrow T_{1x} \Gamma$) is satisfied for every $x \in X$ (here 1 is just another name for the groupoid unit bisection $u^F : X \hookrightarrow \Gamma$) then we call η *unital*. Otherwise said, η is unital whenever the following composite morphism of differentiable vector bundles over X

$$TX \cong (s \circ 1)^*TX \cong 1^*s^*TX \xrightarrow{1^*\eta} 1^*T\Gamma$$

equals $1_* : TX \rightarrow 1^*T\Gamma$, the unique morphism corresponding to $T1 : TX \rightarrow T\Gamma$ by virtue of the pullback universal property. An easy argument involving partitions of unity

shows that any differentiable groupoid which is Hausdorff and second countable admits unital connections.

Let $\Gamma \rightrightarrows X$ be a differentiable groupoid. We shall let $\text{Conn}^k(\Gamma)$ denote the space of connections of class C^k ($k = 0, 1, 2, \dots, \infty$) on $\Gamma \rightrightarrows X$, that is, the affine subspace of $\Gamma^k(\Gamma; L(s^*TX, T\Gamma))$ formed by all those global cross-sections η of class C^k of the real differentiable vector bundle $L(s^*TX, T\Gamma)$ (over the manifold Γ) which are solutions for the equation $s_* \circ \eta = id_{s^*TX}$. We shall moreover let $\text{Conn}_u^k(\Gamma)$ denote the subset of $\text{Conn}^k(\Gamma)$ formed by all unital connections.

By a *pseudo-representation* of class C^k of a differentiable groupoid $\Gamma \rightrightarrows X$ on a (real or complex) differentiable vector bundle E over X we shall mean a morphism of class C^k (between differentiable vector bundles over the manifold Γ) from s^*E into t^*E , in other words, a global cross-section of class C^k of the differentiable vector bundle $L(s^*E, t^*E)$ (over Γ). To each arrow $g \in \Gamma$, a pseudo-representation $\lambda : s^*E \rightarrow t^*E$ assigns a linear map $\lambda_g : E_{sg} \rightarrow E_{tg}$ between the fibers of E corresponding to the source and to the target of g . If λ_g is for each $g \in \Gamma$ a linear isomorphism of E_{sg} onto E_{tg} , we shall say that λ is *invertible*. If $\lambda_{1x} = id_{E_x}$ for all $x \in X$, we shall call λ *unital*. If λ is unital and $\lambda_{g'g} = \lambda_{g'} \circ \lambda_g$ for every composable pair of arrows $(g', g) \in \Gamma_{s \times_t \Gamma}$, we shall call λ *multiplicative* or a *representation*. We shall write $\text{Ps}^k(\Gamma; E)$ for $\Gamma^k(\Gamma; L(s^*E, t^*E))$, the space of pseudo-representations of class C^k of $\Gamma \rightrightarrows X$ on E .

For any connection $H \subset T\Gamma$ of class C^k on a given differentiable groupoid $\Gamma \rightrightarrows X$ we can compose the horizontal lift $\eta^H : s^*TX \rightarrow T\Gamma$ with the vector bundle morphism $t_* : T\Gamma \rightarrow t^*TX$ that (in the way already described) corresponds to the groupoid target mapping $t = t^\Gamma$, thus obtaining a pseudo-representation $\lambda^H : s^*TX \rightarrow t^*TX$ of class C^k of $\Gamma \rightrightarrows X$ on its own base tangent bundle TX . We shall call λ^H the *effect* of H . By an *effective* connection we shall mean one whose effect is a representation.

2. Multiplicative connections

If $X \xrightarrow{f} B \xleftarrow{g} Y$ are differentiable mappings which are transversal then the same must be true of their tangent mappings $TX \xrightarrow{Tf} TB \xleftarrow{Tg} TY$. The differentiable submanifold $TX_{Tf \times Tg} TY \subset TX \times TY$ inherits the structure of a differentiable vector bundle over $X_{f \times_g} Y \subset X \times Y$ from the differentiable vector bundle $TX \times TY = pr_X^*TX \oplus pr_Y^*TY$ (pr_X and pr_Y here denote the two projections $X \leftarrow X \times Y \rightarrow Y$, respectively), because it corresponds under the canonical identification of vector bundles $TX \times TY = T(X \times Y)$ to the subbundle $T(X_{f \times_g} Y) \subset T(X \times Y)$. The induced identification of vector bundles

$$TX_{Tf \times Tg} TY = T(X_{f \times_g} Y) \quad (1)$$

admits the following intuitive description. For any pair of tangent vectors $v \in T_xX$, $w \in T_yY$ such that $(T_xf)(v) = (T_yg)(w) \in T_{f(x)=g(y)}B$ one can by transversality find two differentiable paths $\alpha : \mathbb{R} \rightarrow X$ and $\beta : \mathbb{R} \rightarrow Y$ with $\alpha(0) = x$, $\dot{\alpha}(0) = v$ and $\beta(0) = y$, $\dot{\beta}(0) = w$ such that $f(\alpha(t)) = g(\beta(t))$ for all $t \in \mathbb{R}$. The path $(f, g) : \mathbb{R} \rightarrow X_{f \times_g} Y$ then represents the vector, tangent to $X_{f \times_g} Y$ at (x, y) , corresponding to (v, w) under (1).

Let $\Gamma \rightrightarrows X$ be an arbitrary differentiable groupoid. Its *tangent groupoid* $T\Gamma \rightrightarrows TX$ has source $s^{T\Gamma} := Ts^\Gamma$, target $t^{T\Gamma} := Tt^\Gamma$, composition law $T\Gamma_{Ts \times Tt} T\Gamma \stackrel{(1)}{=} T(\Gamma_{s \times_t \Gamma})$

$\Gamma \xrightarrow{Tm} T\Gamma$, unit $u^{T\Gamma} := Tu^\Gamma$, and inverse $i^{T\Gamma} := Ti^\Gamma$. The tangent composition law $m^{T\Gamma} := Tm^\Gamma \circ (1)$ can be given a more explicit definition, which makes it evident that the structure just introduced satisfies the algebraic axioms for a groupoid and therefore constitutes a differentiable groupoid. Namely, let $w' \in T_{g'}\Gamma$ and $w \in T_g\Gamma$ satisfy $(T_{g'}s)(w') = (T_g t)(w)$, where $sg' = tg$. Choose any two C^∞ paths $\gamma, \gamma' : \mathbb{R} \rightarrow \Gamma$ with $\gamma(0) = g$, $\dot{\gamma}(0) = w$, $\gamma'(0) = g'$ and $\dot{\gamma}'(0) = w'$ such that $s\gamma'(\tau) = t\gamma(\tau)$ for all $\tau \in \mathbb{R}$. Then

$$w'w = \left. \frac{d}{d\tau} \right|_{\tau=0} \gamma'(\tau)\gamma(\tau) \in T_{\gamma'(0)\gamma(0)=g'g}\Gamma.$$

Clearly, any homomorphism of differentiable groupoids $\phi : \Gamma \rightarrow \Delta$ induces a *tangent homomorphism* $T\phi : T\Gamma \rightarrow T\Delta$. We thus obtain a *tangent functor* $T(-)$ from the category of differentiable groupoids into itself.

Let $\Gamma \rightrightarrows X$ be an arbitrary differentiable groupoid. As in the preceding section, we let $T^\uparrow\Gamma \subset T\Gamma$ denote the s -vertical subbundle. The *algebroid bundle* of $\Gamma \rightrightarrows X$ is defined to be the differentiable vector bundle over X given by $\text{alg } \Gamma := 1^*T^\uparrow\Gamma$ (pullback along $1 = u : X \hookrightarrow \Gamma$). Let us put $\mathfrak{g} = \text{alg } \Gamma$ for brevity. For each arrow $g \in \Gamma$ the right translation mapping $\tau_{g^{-1}} : \Gamma^{sg} \xrightarrow{\sim} \Gamma^{tg}$ ($h \mapsto hg^{-1}$) is a diffeomorphism which makes g correspond to the unit 1_{tg} . Taking its differential at g , we obtain an isomorphism of tangent spaces $T_g\tau_{g^{-1}} : T_g\Gamma^{sg} \xrightarrow{\sim} T_{1_{tg}}\Gamma^{tg}$. The various linear maps $\omega_g^\Gamma := T_g\tau_{g^{-1}}$ (as g ranges over Γ) fit together into an isomorphism of differentiable vector bundles over Γ

$$\omega^\Gamma : T^\uparrow\Gamma \xrightarrow{\sim} t^*\mathfrak{g},$$

which is usually referred to as the *Maurer–Cartan isomorphism* (or *Maurer–Cartan form*) associated to Γ .

A connection $H \subset T\Gamma$ on a differentiable groupoid $\Gamma \rightrightarrows X$ is said to be *multiplicative* if $H \rightrightarrows TX$ constitutes a subgroupoid (by necessity, over the whole of TX) of the tangent groupoid $T\Gamma \rightrightarrows TX$. Trivially, multiplicative connections are unital. They are also always effective, as we will see presently. In order to minimize notational clutter in the discussion to follow, we will resort to the simplicial notation for the manifold of composable arrows of a differentiable groupoid. We seize the opportunity to set some overall conventions.

For each integer $k \geq 2$, let $\Gamma_{(k)}$ denote the differentiable submanifold of the k -fold Cartesian product $\Gamma^{\times k} := \Gamma \times \cdots \times \Gamma$ formed by all composable k -tuples of arrows:

$$\Gamma_{(k)} \stackrel{\text{def}}{=} \{(g_1, g_2, \dots, g_k) \in \Gamma^{\times k} \mid sg_{i-1} = tg_i \text{ for every } i = 2, \dots, k\}.$$

Thus, $\Gamma_{(2)} := \Gamma \times_s \Gamma$. We complete the series by setting $\Gamma_{(0)} := X$ and $\Gamma_{(1)} := \Gamma$. We let $X \xleftarrow{s_{(k)}} \Gamma_{(k)} \xrightarrow{t_{(k)}} X$ denote the two mappings given respectively by $(g_1, g_2, \dots, g_k) \mapsto sg_k$ and $\mapsto tg_1$. For completeness, we also set $s_{(1)} := s$, $t_{(1)} := t$, and $t_{(0)} := s_{(0)} := \text{id}_X$. We will occasionally allow the abridged versions $\Gamma_k := \Gamma_{(k)}$ etc. of the (official) notations just introduced, but only in such situations when no ambiguity with our notation for target fibers $\Gamma_x := \Gamma(-, x)$ is likely to arise.

Recall that for an arbitrary connection $H \subset T\Gamma$ on a given differentiable groupoid $\Gamma \rightrightarrows X$ we have the associated horizontal lift $\eta^H : s^*TX \rightarrow T\Gamma$ and pseudo-representation $\lambda^H : s^*TX \rightarrow t^*TX$. Letting $pr_1, pr_2, m : \Gamma_{(2)} \rightarrow \Gamma$ respectively denote the

1st projection, the 2nd projection and the groupoid's composition law, we form the following three morphisms of differentiable vector bundles over $\Gamma_{(2)}$.

$$\begin{aligned} s_{(2)}^*TX &\cong pr_2^*s^*TX \xrightarrow{pr_2^*\lambda^H} pr_2^*t^*TX \cong pr_1^*s^*TX \xrightarrow{pr_1^*\eta^H} pr_1^*T\Gamma \\ s_{(2)}^*TX &\cong pr_2^*s^*TX \xrightarrow{pr_2^*\eta^H} pr_2^*T\Gamma \\ s_{(2)}^*TX &\cong m^*s^*TX \xrightarrow{m^*\eta^H} m^*T\Gamma \end{aligned}$$

The first two of them can be combined into a single morphism, say, $\alpha = (pr_1^*\eta^H \circ pr_2^*\lambda^H, pr_2^*\eta^H) : s_{(2)}^*TX \rightarrow pr_1^*T\Gamma \oplus pr_2^*T\Gamma$, which is easily recognized to factor through the subbundle $T\Gamma_{T_s \times T_t} T\Gamma \subset pr_1^*T\Gamma \oplus pr_2^*T\Gamma$ as in the commutative diagram below.

$$\begin{array}{ccc} s_{(2)}^*TX & \xrightarrow{\alpha} & pr_1^*T\Gamma \oplus pr_2^*T\Gamma = T(\Gamma \times \Gamma) | \Gamma_{(2)} \\ & \searrow & \uparrow \\ & & T\Gamma_{T_s \times T_t} T\Gamma \xrightarrow{(1)} T(\Gamma_s \times_t \Gamma) \xrightarrow{m_*} m^*T\Gamma \end{array}$$

The “basic curvature” of H (cf. [1], Subsection 2.4), here denoted by R^H , is defined to be the morphism of differentiable vector bundles over $\Gamma_{(2)}$ that results from the expression

$$R^H \stackrel{\text{def}}{=} m^*\omega^\Gamma \circ (m^*\eta^H - m_* \circ (pr_1^*\eta^H \circ pr_2^*\lambda^H, pr_2^*\eta^H)) \in \text{Hom}_{\Gamma_{(2)}}(s_{(2)}^*TX, t_{(2)}^*\mathfrak{g}).$$

Explicitly, at every point $(g', g) \in \Gamma_{(2)}$ the basic curvature $R_{g',g}^H$ is given by the linear map

$$T_{sg}X \ni v \mapsto R_{g',g}^H v \stackrel{\text{def}}{=} \omega_{g',g}^\Gamma(\eta_{g',g}^H v - (\eta_{g'}^H \lambda_g^H v) \eta_g^H v) \in \mathfrak{g}_{t(g'g)=tg'}.$$

Proposition 2.1. *The following properties are equivalent for an arbitrary unital connection $H \subset T\Gamma$ on a differentiable groupoid $\Gamma \rightrightarrows X$.*

- (a) H is multiplicative.
- (b) The identity $\eta_{g',g}^H v = (\eta_{g'}^H \lambda_g^H v) \eta_g^H v$ holds for every composable pair of arrows (g', g) for all tangent vectors $v \in T_{sg}X$.
- (c) $R^H = 0$.

The proof is straightforward. Observe that under the assumption that the linear endomorphism $\lambda_{1x}^H \in \text{End}(T_x X)$ is surjective (equivalently, injective or bijective) the property (b) alone is sufficient for the unitality of H at any given point $x \in X$. Indeed, if that property holds then $\eta_{1x}^H v = \eta_{1x1x}^H v = (\eta_{1x}^H \lambda_{1x}^H v) \eta_{1x}^H v$ and hence, dividing out by $\eta_{1x}^H v$ on the right within $T\Gamma \rightrightarrows TX$ for variable $v \in T_x X$, the identity of linear maps $T_x 1 \circ \lambda_{1x}^H = \eta_{1x}^H \circ \lambda_{1x}^H$ drops out, which, if λ_{1x}^H is cancellable, yields the desired conclusion. However, it is very easy to construct examples of non-unital connections with the aforesaid property. So, that property does not imply unitality in general.

Corollary 2.2. *Multiplicative connections are effective.*

Proof. Apply $T_{g'g}t$ to both members of the identity $\eta_{g',g}^H v = (\eta_{g'}^H \lambda_g^H v) \eta_g^H v$. □

A first glance at multiplicative connections through examples

In order to get some feeling for multiplicative connections, we start looking into the simple case of action groupoids, which is already instructive. We are going to provide examples of complete classifications of multiplicative connections for a few of such groupoids. The classification will be achieved by ad hoc, elementary means, mainly by direct computation. We hope to come back to the classification problem with a more systematic treatment at some later point.

Let G be an arbitrary Lie group. The tangent multiplication law $m^{TG} : TG \times TG \rightarrow TG$ [when regarded as a vector bundle morphism $m_*^{TG} : TG \times TG \rightarrow (m^G)^*TG$] fibers over the manifold of all pairs of group elements $g, h \in G$ into linear maps $m_{g,h}^{TG}$ given by

$$T_g G \oplus T_h G = T_{(g,h)}(G \times G) \xrightarrow{T_{(g,h)}m} T_{gh} G, (u, v) \mapsto uv.$$

As before, let $\tau_g : G \xrightarrow{\sim} G$ denote the right translation by a group element $g \in G$, that is, the mapping of G into itself given by $x \mapsto xg$. Also let $c_g : G \xrightarrow{\sim} G$ denote the conjugation by g , that is, the mapping $x \mapsto gxg^{-1}$. We have the following commutative diagram of smooth mappings.

$$\begin{array}{ccc} (x, y) & & G \times G \xrightarrow{m} G \\ \downarrow & & \downarrow \tau_g^{-1} \times (c_g \circ \tau_h^{-1}) \quad \downarrow \tau_{gh}^{-1} \\ (xg^{-1}, gyh^{-1}g^{-1}) & & G \times G \xrightarrow{m} G \end{array} \quad (2)$$

Recalling our definitions, according to which $T_g \tau_g^{-1} = T_g \tau_{g^{-1}} : T_g G \xrightarrow{\sim} T_1 G$ is the Maurer–Cartan isomorphism $\omega_g : T_g G \xrightarrow{\sim} \mathfrak{g} = \text{alg } G (= T_1 G)$ at the point $g \in G$, upon differentiating the diagram (2) at $(g, h) \in G \times G$ we obtain the following identity of Lie algebra valued linear maps

$$\omega_{gh} \circ T_{(g,h)}m = T_{(1,1)}m \circ (\omega_g \times [\text{Ad}_G(g) \circ \omega_h]),$$

where $\text{Ad}_G : G \rightarrow GL(\mathfrak{g})$ denotes the adjoint representation of G . Since the linear map $T_{(1,1)}m : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g}$ is nothing but the sum operation $(X, Y) \mapsto X + Y$, we conclude that

$$m_{g,h}^{TG} = \omega_{gh}^{-1} \circ (\omega_g \circ pr_I + \text{Ad}_G(g) \circ \omega_h \circ pr_{II}). \quad (3)$$

Next, suppose we are given some smooth (left) action $a : G \times U \rightarrow U$ of our Lie group G on a smooth manifold U . Consider the action groupoid $G \ltimes U := G \times U \xrightarrow[pr_U]{a} U$, which is of course a Lie groupoid. Let H be an arbitrary connection on $G \ltimes U$. Let $\chi^H : pr_U^* TU \rightarrow pr_G^* TG$ denote the morphism of smooth vector bundles over $G \times U$ obtained by composition of η^H with $(pr_G)_*$:

$$\chi^H \stackrel{\text{def}}{=} pr_U^* TU \xrightarrow{\eta^H} T(G \times U) \xrightarrow{(pr_G)_*} pr_G^* TG.$$

(Of course, $pr_G : G \times U \rightarrow G$ here denotes the projection on G .) This morphism fibers over $G \times U$ into the linear maps $\chi_{g,u}^H := T_{(g,u)}pr_G \circ \eta_{g,u}^H : T_u U \rightarrow T_g G$. We can further

compose χ^H with the Maurer–Cartan form $\omega : TG \rightarrow \underline{\mathfrak{g}}_G := G \times \mathfrak{g}$ (= trivial vector bundle over G with fiber \mathfrak{g}) to obtain a vector bundle morphism

$$X^H \stackrel{\text{def}}{=} pr_G^* \omega \circ \chi^H : pr_U^* TU \rightarrow pr_G^* \underline{\mathfrak{g}}_G = \underline{\mathfrak{g}}_{G \times U}, \quad (4)$$

whose fiber at each point $(g, u) \in G \times U$ is a linear map $X_{g,u}^H := \omega_g \circ \chi_{g,u}^H : T_u U \rightarrow \mathfrak{g}$ with values in the Lie algebra of G . It is clear that if we are given an arbitrary smooth vector bundle morphism $X : pr_U^* TU \rightarrow \underline{\mathfrak{g}}_{G \times U}$ then there exists exactly one connection H on the action groupoid $G \ltimes U$ for which $X = X^H$.

We proceed to derive a system of equations for the linear maps $X_{g,u}^H$ which is to express the condition of multiplicativity for the connection H in a form particularly suitable for computations. In the present context, the identity 2.1(b) reads

$$\eta_{gh,u}^H v = (\eta_{g,hu}^H \lambda_{h,u}^H v) \eta_{h,u}^H v.$$

Upon applying $T_{(gh,u)} pr_G : T_{(gh,u)}(G \times U) \rightarrow T_{gh}G$ to both sides of this identity,

$$\begin{aligned} \chi_{gh,u}^H v &= [T_{(gh,u)} pr_G \circ T_{(g,hu;h,u)} m^{G \ltimes U}] (\eta_{g,hu}^H \lambda_{h,u}^H v, \eta_{h,u}^H v) \\ &= [T_{(g,h)} m^G \circ (T_{(g,hu)} pr_G \times T_{(h,u)} pr_G)] (\eta_{g,hu}^H \lambda_{h,u}^H v, \eta_{h,u}^H v) \\ &= (m_{g,h}^{TG}) (\chi_{g,hu}^H \lambda_{h,u}^H v, \chi_{h,u}^H v) \\ &= \omega_{gh}^{-1} (\omega_g (\chi_{g,hu}^H \lambda_{h,u}^H v) + [\text{Ad}_G(g) \circ \omega_h] (\chi_{h,u}^H v)) \quad [\text{by (3)}] \\ &= \omega_{gh}^{-1} (X_{g,hu}^H \lambda_{h,u}^H v + \text{Ad}_G(g) X_{h,u}^H v) \end{aligned} \quad (5)$$

we see that the condition 2.1(b) can be reformulated as a system of cocycle equations for the linear maps $\{X_{g,u}^H\}$, namely, for all $g, h \in G$ and all $u \in U$:

$$\boxed{\text{Ad}_G(g) \circ X_{h,u}^H - X_{gh,u}^H + X_{g,hu}^H \circ \lambda_{h,u}^H = 0.} \quad (6a)$$

The condition of unitality for H is expressed by the equations

$$\boxed{X_{1,u}^H = 0.} \quad (6b)$$

We observe that the zero morphism is always trivially a solution for the multiplicativity equations (6). Hence the connection Φ on $G \ltimes U \rightrightarrows U$ characterized by the condition $X^\Phi = 0$ is always multiplicative. We therefore see that action groupoids always admit multiplicative connections. However, more general Lie groupoids may not admit any; see [25] for a counterexample. As a matter of fact, the existence of a *flat* (i.e., integrable as a distribution) multiplicative connection is a rather strong requirement; compare [5, Corollary 3.12].

Example A: Torus bundles

Let us next restrict our attention to the case of a trivial G -action: $a = pr_U$. In such case, the action groupoid $\Gamma = G \ltimes U \xrightarrow{s=t} U$ is simply a (trivial) bundle of Lie groups over U with fiber G . For an arbitrary groupoid connection H on Γ , the corresponding pseudo-representation λ^H is necessarily trivial: $\lambda_{g,u}^H = id_{T_u U}$ for every $(g, u) \in G \times U$.

In particular, we see that every groupoid connection H on Γ is effective. Because of this, in view of the comments preceding Corollary 2.2, a groupoid connection H on Γ is multiplicative if, and only if, it satisfies the equations (5) or, equivalently, the equations (6a), which in the present context take the following shape, respectively.

$$\begin{aligned}\chi_{gh,u}^H v &= (m_{g,h}^{TG})(\chi_{g,u}^H v, \chi_{h,u}^H v) \\ \text{Ad}_G(g) \circ X_{h,u}^H - X_{gh,u}^H + X_{g,u}^H &= 0\end{aligned}\tag{7}$$

If for any given tangent vector $v \in T_u U$ we put $\zeta_v^H(g) := \chi_{g,u}^H v$ for variable $g \in G$, we obtain a differentiable cross-section $\zeta_v^H \in \Gamma^\infty(G; TG)$, in other words, a C^∞ vector field ζ_v^H on G . If H is multiplicative then, in virtue of the equation (7), the vector field ζ_v^H must satisfy the following condition for all $g, h \in G$.

$$\zeta_v^H(gh) = \zeta_v^H(g)\zeta_v^H(h)$$

In general, we shall say that a vector field $\zeta : G \rightarrow TG$ on a Lie group G is *multiplicative* if the identity $\zeta(gh) = \zeta(g)\zeta(h)$ holds for all $g, h \in G$. Rephrasing (7), we can say that our connection H is multiplicative if, and only if, ζ_v^H is a multiplicative vector field on G for each tangent vector $v \in TU$. The study of multiplicative connections on the trivial Lie group bundle $\Gamma \xrightarrow{s=l} U$ therefore reduces in principle to the study of multiplicative vector fields on G .

For the purpose of analyzing multiplicative vector fields on G , it will be convenient, for any given vector field $\zeta \in \Gamma(G; TG)$, to let $Z_\zeta : G \rightarrow \mathfrak{g}$ denote the Lie algebra valued function on G given by $Z_\zeta(g) := \omega_g(\zeta(g))$. For instance, for $\zeta = \zeta_v^H$ as in the preceding paragraph, we have $Z_v^H(g) := Z_{\zeta_v^H}(g) = \omega_g(\chi_{g,u}^H v) = X_{g,u}^H v$. By the formula (3), a vector field $\zeta : G \rightarrow TG$ is multiplicative if, and only if, the corresponding function $Z_\zeta : G \rightarrow \mathfrak{g}$ is a 1-cocycle for the Lie group cohomology of G with coefficients in the adjoint representation $\text{Ad}_G : G \rightarrow GL(\mathfrak{g})$. In simpler words, the condition that ζ is multiplicative is expressed by the following formula:

$$\text{Ad}_G(g)Z_\zeta(h) - Z_\zeta(gh) + Z_\zeta(g) = 0.$$

If our Lie group G is commutative then $\text{Ad}_G = \text{id}$ will be the trivial representation and hence a vector field ζ on G will be multiplicative if and only if Z_ζ is additive: $Z_\zeta(gh) = Z_\zeta(g) + Z_\zeta(h)$ for all $g, h \in G$. Now if $\alpha : \mathbb{R} \rightarrow G$ is an arbitrary one-parameter subgroup of G then the composition $Z_\zeta \circ \alpha$ is a continuous additive map of \mathbb{R} into \mathfrak{g} and hence is an \mathbb{R} -linear map. If the image of α is a compact subgroup of G then the image of this subgroup under Z_ζ is a compact linear subspace of \mathfrak{g} and hence is necessarily equal to $\{0\}$. We conclude that under the assumption that ζ is multiplicative one has $Z_\zeta(g) = 0$ for every $g \in G$ lying on a compact one-parameter subgroup of G .

If the Lie group G is compact (besides being commutative) then the set of all those group elements that lie on some compact one-parameter subgroup of G will be dense within the identity component G_0 of G . Hence the function Z_ζ will vanish identically over G_0 for any multiplicative vector field ζ on G . Being additive, Z_ζ will then take only finitely many values, one for each connected component of G . But the image of Z_ζ is also a \mathbb{Z} -sublattice of \mathfrak{g} and therefore can only be zero. We conclude that on a compact commutative Lie group the only multiplicative vector field is the zero vector field.

2.3. For the trivial action $a = pr_U : G \times U \rightarrow U$ of a compact abelian Lie group G on a smooth manifold U the only multiplicative connection on the associated action groupoid $G \ltimes U \xrightarrow{s=t} U$ is the connection Φ characterized by the condition $X^\Phi = 0$. More generally, any locally trivial bundle of compact abelian Lie groups $\Gamma \xrightarrow{s=t} M$ over a smooth manifold M admits exactly one multiplicative connection.

Example B: Homogeneous circle-spaces

At the other extreme, there is the case of a transitive Lie group action. Among the simplest examples there are the transitive actions of the circle group $SO(2)$. Namely, let $SO(2) \times M \rightarrow M$ be an arbitrary transitive smooth (left) action of $SO(2)$ on a smooth manifold M . One calls this a *homogeneous $SO(2)$ -space*. Of course, M is necessarily connected and compact. The choice of a base point $x_0 \in M$ defines, setting for brevity $G = SO(2)$ and letting $K = \text{Stab}_G(x_0)$ denote the stabilizer subgroup at x_0 , a G -equivariant diffeomorphism $G/K \xrightarrow{\sim} M$, $gK \mapsto gx_0$ of the (left) G -space G/K of right K -cosets of G onto M . Ruling out the case $K = G$ as plainly uninteresting, we may and will assume that K is zero-dimensional (as a submanifold of G) and, therefore, discrete. Since K is also a closed (hence compact) subgroup of G , it must actually be finite. In conclusion, our action groupoid $G \ltimes M$ is isomorphic (as a Lie groupoid) to the standard coset-action groupoid $G \ltimes G/K$ for some finite subgroup $K \subset G$ of order, say, k . For our purposes, it will not be restrictive to assume that $G \ltimes M$ really is $G \ltimes G/K$.

The action groupoid $G \ltimes G/K$ is isomorphic to the action groupoid $G \ltimes^k G$ associated with the twisted (left) translation action $G \times G \rightarrow G$, $(z, u) \mapsto z^k u$. If we identify $SO(2)$ with the group of complex numbers of modulus one under complex multiplication, the exponential $\theta \mapsto \exp 2\pi i \theta$ will be a Lie group homomorphism from the additive group of the real numbers $(\mathbb{R}, +)$ onto $SO(2)$. The same map will be $\exp(2\pi i -)$ -equivariant with respect to the action of $(\mathbb{R}, +)$ on itself given by $(\theta, a) \mapsto k\theta + a$ and with respect to the above k -twisted self-action of $SO(2)$. It will therefore promote to a Lie groupoid homomorphism, say, ε between the corresponding action groupoids:

$$\varepsilon : \mathbb{R} \ltimes^k \mathbb{R} \longrightarrow SO(2) \ltimes^k SO(2), (\theta, a) \mapsto (\exp 2\pi i \theta, \exp 2\pi i a).$$

Since the tangent bundle of the real line is canonically trivial, one will have a canonical identification between, on the one side, the vector bundle morphisms of type (4) for the action groupoid $\mathbb{R} \ltimes^k \mathbb{R}$ (that is to say, the connections on that action groupoid) and, on the other side, the real functions of class C^∞ on \mathbb{R}^2 . The connection on $\mathbb{R} \ltimes^k \mathbb{R}$ corresponding in this way to a C^∞ function $X : \mathbb{R}^2 \rightarrow \mathbb{R}$ will be related through ε to some connection on $SO(2) \ltimes^k SO(2)$ precisely when the function X is $\mathbb{Z} \times \mathbb{Z}$ -periodic, that is, when $X(\theta + m, a + n) = X(\theta, a)$ for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$.

The canonical triviality of the tangent bundle $T\mathbb{R}$ allows one to canonically identify the pseudo-representations of the groupoid $\mathbb{R} \ltimes^k \mathbb{R}$ on the vector bundle $T\mathbb{R}$ with the functions on the real plane. The pseudo-representation corresponding to a function $\Lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a representation if, and only if, Λ obeys the following constraints.

$$\Lambda(\theta' + \theta, a) = \Lambda(\theta', k\theta + a)\Lambda(\theta, a) \tag{8a}$$

$$\Lambda(0, a) = 1 \tag{8b}$$

The pseudo-representation associated to the connection on $\mathbb{R} \ltimes^k \mathbb{R}$ corresponding to a given function X , in particular, corresponds to the function Λ^X on \mathbb{R}^2 given by

$$\Lambda^X(\theta, a) = 1 + kX(\theta, a). \quad (9)$$

When rephrased in terms of X , the condition of multiplicativity for the connection corresponding to X takes the following form, the adjoint representation being now trivial.

$$X(\theta' + \theta, a) = X(\theta, a) + X(\theta', k\theta + a)[1 + kX(\theta, a)] \quad (10a)$$

$$X(0, a) = 0 \quad (10b)$$

We notice that these equations are nothing but the equations (8) written out for $\Lambda = \Lambda^X$ given by (9). Thus, we see that an arbitrary connection on $\mathbb{R} \ltimes^k \mathbb{R}$ [and, consequently, on $SO(2) \ltimes^k SO(2)$] is multiplicative if, and only if, it is effective. We also see that the connection on $\mathbb{R} \ltimes^k \mathbb{R}$ [or on $SO(2) \ltimes^k SO(2)$] that corresponds to the constant function $X(\theta, a) = -1/k$, which is a function satisfying (10a) but not (10b), is a connection that is non-unital but nevertheless satisfies the condition 2.1(b). The associated pseudo-representation vanishes identically.

Given an arbitrary C^∞ real function X on \mathbb{R}^2 which satisfies the equations (10), let $f(\theta) = X(\theta, 0)$ denote its restriction to the x -axis. This is a C^∞ real function on \mathbb{R} which satisfies the condition $f(0) = 0$ and which never takes the value $-1/k$ [if it did, say, if $f(\theta) = -1/k$ for some θ , then from (10a) with $a = 0$ and $\theta' = -\theta$ we would deduce the absurdity $0 = f(0) = f(\theta) = -1/k$]. It completely determines X via the equation

$$X(\theta, a) = \frac{f(\theta + a/k) - f(a/k)}{1 + kf(a/k)}, \quad (11)$$

which follows at once from (10a) by first setting $a = 0$ in that equation and then substituting back a/k for θ and θ for θ' in the resulting expression. Conversely, given an arbitrary real function f on \mathbb{R} of class C^∞ such that $f(\theta) \neq -1/k$ for all θ and such that $f(0) = 0$, one can define a function X on \mathbb{R}^2 by means of (11). This will be a C^∞ function satisfying the equations (10), as one can easily check. Moreover, $X(\theta, 0)$ will be equal to $f(\theta)$ for all θ . In conclusion, the equation (11) defines a parameterization of the set of all multiplicative connections on the action groupoid $\mathbb{R} \ltimes^k \mathbb{R}$ by the set of all functions $f \in C^\infty(\mathbb{R},]-1/k, \infty[)$ that vanish at zero.

It remains to be seen when exactly a function $f \in C^\infty(\mathbb{R},]-1/k, \infty[)$ with $f(0) = 0$ corresponds to a $\mathbb{Z} \times \mathbb{Z}$ -periodic function under the above parameterization. In one direction, it is clear that the function X that corresponds to f is $\mathbb{Z} \times \mathbb{Z}$ -periodic if f is $k^{-1}\mathbb{Z}$ -periodic. We claim that this condition is also necessary. Indeed, if X is $\mathbb{Z} \times \mathbb{Z}$ -periodic then in particular $X(\theta, 1) = X(\theta, 0)$ for all $\theta \in \mathbb{R}$, which amounts to the identity

$$f(\theta + 1/k) - f(\theta) = f(1/k)[1 + kf(\theta)].$$

Now, if $f(1/k) \neq 0$ then the sequence of real numbers

$$0 = f(0), f(1/k), \dots, f((k-1)/k), f(k/k) = f(1)$$

must be strictly increasing or decreasing according to the sign of $f(1/k) \gtrless 0$ because $1 + kf(\theta)$ is always positive. Hence $f(1) \gtrless 0$. But this is impossible because $f(1) = X(1, 0) = X(0, 0) = 0$. Therefore, it actually was $f(1/k) = 0$, and consequently $f(\theta + 1/k) = f(\theta)$ for all $\theta \in \mathbb{R}$.

2.4. *The set of all multiplicative connections on an arbitrary positive-dimensional homogeneous circle-space $SO(2) \ltimes M$ coincides with the set of all effective connections and is canonically parameterized by the convex subset of $C^\infty(S^1, \mathbb{R})$ consisting of all those functions that vanish at 1 and never take the value $-1/k$, where $k \in \mathbb{N}_{\geq 1}$ is the order of any stabilizer subgroup for the given action of $SO(2)$ on M . Thus, any two multiplicative connections on $SO(2) \ltimes M$ can be smoothly deformed into each other through multiplicative connections.*

3. The averaging operator

In the present section and in the next three the only differentiable groupoids which we shall be considering will be those that are simultaneously *Lie* and *proper*. Our goal, in this section, is to show the equivalence of the following two conditions for any such groupoid, say, Γ : (i) Γ admits an effective connection; (ii) Γ admits a multiplicative connection. We shall achieve this by means of an averaging technique. As a matter of fact, the averaging technique in question was originally inspired to us by the deformation argument used in [6] to give a new proof of the linearization theorem for proper Lie groupoids [30, 31]; the reader will find many analogies between the subsections 2.3 to 2.5 of [6] and our exposition. Before embarking on a discussion of our averaging method, we shall introduce some auxiliary notations and make a couple of general remarks. Let us now once and for all assume that we are given a proper Lie groupoid $\Gamma \rightrightarrows M$; this will remain fixed throughout.

Let Γ_{\div} denote the differentiable submanifold of $\Gamma \times \Gamma$ formed by all the *divisible* pairs of arrows: $\Gamma_{\div} := \Gamma \times_s \Gamma = \{(g, h) \in \Gamma \times \Gamma \mid sg = sh\}$. Let $q_{\div} : \Gamma_{\div} \rightarrow \Gamma$ denote the mapping given by $(g, h) \mapsto gh^{-1}$, to be called the *quotient* or *ratio*, and let l_{\div} and r_{\div} denote the left and the right projection from Γ_{\div} onto Γ . Furthermore let s_{\div} and t_{\div} denote the two mappings of Γ_{\div} onto M given the first one by $s_{\div}(g, h) := s(gh^{-1}) = th$ and the second one by $t_{\div}(g, h) := t(gh^{-1}) = tg$. We adopt the notation $w_1 \div w_2$ for the quotient $w_1 w_2^{-1}$ of a divisible pair of tangent vectors $(w_1, w_2) \in T\Gamma_{Ts \times Ts} T\Gamma$ within the tangent groupoid $T\Gamma \rightrightarrows TM$. (The main reason why we find this notation convenient is that it avoids clutter in the formation of superscripts.) We observe that the operation $(w_1, w_2) \mapsto w_1 \div w_2$ can be expressed as the following composite mapping

$$T\Gamma_{Ts \times Ts} T\Gamma \stackrel{(1)}{=} T(\Gamma \times_s \Gamma) = T(\Gamma_{\div}) \xrightarrow{Tq_{\div}} T\Gamma \quad (12)$$

where (1) indicates the isomorphism of differentiable vector bundles over Γ_{\div} that was discussed in a more abstract setting at the beginning of Section 2; as in that section, $T\Gamma_{Ts \times Ts} T\Gamma$ is given the vector bundle structure (over the manifold Γ_{\div}) that makes it into a subbundle of $l_{\div}^* T\Gamma \oplus r_{\div}^* T\Gamma$.

By Proposition 2.1, for any connection H on $\Gamma \rightrightarrows M$ (assuming this is unital, or at least that the associated pseudo-representation λ^H on TM is unital) the condition

$$[\forall (g', g) \in \Gamma \times_t \Gamma] \quad \eta_{g'g}^H = (\eta_{g'}^H \circ \lambda_g^H) \eta_g^H : T_{sg} M \rightarrow T_{g'g} \Gamma \quad (13)$$

is equivalent to the multiplicativity of H . Making $g' = g^{-1}$ in the preceding equation, and exploiting the required unitality of H , we get the identity

$$[\forall g \in \Gamma] \quad (\eta_g^H)^{-1} = \eta_{g^{-1}}^H \circ \lambda_g^H.$$

Combining the above equations, and referring to the fact that for each multiplicative connection on $\Gamma \rightrightarrows M$ the corresponding pseudo-representation is necessarily a functor of Γ into $GL(TM)$, we obtain

$$[\forall (g, h) \in \Gamma \times_s \Gamma] \quad \eta_{gh^{-1}}^H \circ \lambda_h^H = \eta_g^H \div \eta_h^H : T_{sg=sh}M \longrightarrow T_{gh^{-1}}\Gamma \quad (14)$$

where $\eta_g^H \div \eta_h^H$ denotes the linear map $T_{sg=sh}M \ni v \mapsto (\eta_g^H v) \div (\eta_h^H v) \in T_{gh^{-1}}\Gamma$.

Lemma 3.1. *Under the assumption that H is a unital connection on $\Gamma \rightrightarrows M$, the condition (14) expresses the multiplicativity of H .*

Proof. To go back to the equation (13) starting from the equation (14) just set $gh^{-1} = h'$ in the latter and then multiply each one of its members by η_h^H on the right. \square

We shall say that a connection H on $\Gamma \rightrightarrows M$ is *non-degenerate*, or *invertible*, if the pseudo-representation λ^H which H determines on the tangent bundle TM is invertible, in other words, if we have $\lambda^H(g) \in \text{Lis}(T_{sg}M, T_{tg}M)$ for all $g \in \Gamma$. The notion we are introducing here is not entirely new; the non-degeneracy condition for a Lie groupoid connection is explicitly written down in Definition 2.2 of [25], although in that work it does not really play a distinguished role since it is implied by other conditions (namely, multiplicativity) which are postulated as part of the same definition. For each value of $p = 0, 1, 2, \dots, \infty$, let

$$\text{Conn}_*^p(\Gamma) \subset \text{Conn}^p(\Gamma)$$

denote the subset traced out by all non-degenerate connections within the Fréchet manifold of all connections of class C^p on $\Gamma \rightrightarrows M$.

Definition 3.2. For any non-degenerate connection H on $\Gamma \rightrightarrows M$, put

$$\delta^H(g, h) \stackrel{\text{def}}{=} (\eta_g^H \div \eta_h^H) \circ (\lambda_h^H)^{-1} \in L(T_{th}M, T_{gh^{-1}}\Gamma) \quad (15a)$$

for every divisible pair $(g, h) \in \Gamma_{\div}$. [As above, $\eta_g^H \div \eta_h^H$ indicates the linear map of $T_{sg=sh}M$ into $T_{gh^{-1}}\Gamma$ given by $v \mapsto (\eta_g^H v) \div (\eta_h^H v)$.] We shall refer to the global vector bundle cross-section defined by (15a) namely to

$$\delta^H \in \Gamma(\Gamma_{\div}; L(s^*TM, q_{\div}^*T\Gamma)) \quad (15b)$$

as the *difference cocycle* associated to H , since in a certain sense this measures for every divisible pair $(g, h) \in \Gamma_{\div}$ the difference—or better, the ratio—between η_g^H and η_h^H relative to the tangent multiplicative structure of $\Gamma \rightrightarrows M$. The sense in which δ^H should be understood to be a cocycle will be made somehow more precise later. For the time being, we limit ourselves to the vague remark that in general it is reasonable to expect the ratio to be a cocycle with respect to multiplication: $\frac{g}{h} \frac{h}{k} = \frac{g}{k}$.

Let us first of all point out that the difference cocycle δ^H is of the same class of differentiability as the connection H that gives rise to it. Indeed, suppose that H is an element of $\text{Conn}_*^p(\Gamma)$ for some value of $p = 0, 1, 2, \dots, \infty$. This means that $\eta^H \in \Gamma^p(\Gamma; L(s^*TM, T\Gamma))$ and thus, in particular, that $\lambda^H \in \Gamma^p(\Gamma; L(s^*TM, t^*TM))$.

It is evident from (15a) that δ^H can be decomposed into the sequence of vector bundle morphisms (over the manifold Γ_{\div}) reproduced below,

$$\begin{aligned} s_{\div}^* TM &\cong r_{\div}^* t^* TM \xrightarrow{r_{\div}^* (\lambda^H)^{-1}} r_{\div}^* s^* TM (\cong l_{\div}^* s^* TM) \\ &\xrightarrow{(l_{\div}^* \eta^H, r_{\div}^* \eta^H)} l_{\div}^* T\Gamma \oplus_{l_{\div}^* s^* TM = r_{\div}^* s^* TM} r_{\div}^* T\Gamma = T\Gamma_{Ts \times Ts} T\Gamma \xrightarrow{(12)} q_{\div}^* T\Gamma \end{aligned}$$

each constituent of which is of class C^p , when not C^∞ .

Making use now for the first time of the hypothesis that our Lie groupoid $\Gamma \rightrightarrows M$ is proper, we fix a normalized left Haar system ν on $\Gamma \rightrightarrows M$. (We know this is always possible in virtue of the propositions B.4 and B.6.) We remind the reader that the left invariance of the system is expressed by the law $\int_{tk=sh} f(hk) d\nu_{sh}(k) = \int_{tk'=th} f(k') d\nu_{th}(k')$.

Definition 3.3. Let H be a non-degenerate connection on $\Gamma \rightrightarrows M$. For each arrow $g \in \Gamma$, let $\hat{\eta}_g^H$ denote the linear map

$$T_{sg}M \ni v \mapsto \hat{\eta}_g^H v \stackrel{\text{def}}{=} \int_{tk=sg} \delta^H(gk, k) v d\nu_{sg}(k) \in T_g\Gamma. \quad (16a)$$

[This expression makes sense because $\delta^H(gk, k)$ is a linear map of $T_{tk=sg}M$ into $T_{gk^{-1}=g}T$ for all $\xleftarrow{g} \xleftarrow{k}$.] We shall refer to the global, vector bundle cross-section

$$\hat{\eta}^H \in \Gamma(\Gamma; L(s^*TM, T\Gamma)) \quad (16b)$$

defined by (16a) as the (*multiplicative*) *average* of H (relative to the chosen normalized left Haar system ν on $\Gamma \rightrightarrows M$).

Lemma 3.4. *The multiplicative average $\hat{\eta}^H$ of any non-degenerate C^p connection $H \in \text{Conn}_*^p(\Gamma)$ is itself the horizontal lift for a unique connection \hat{H} of class C^p on $\Gamma \rightrightarrows M$, which is always unital, and which will itself be called the multiplicative average of H .*

Proof. In the first place, let us address the question of the differentiability of the global cross-section (16b). We already know that δ^H is a C^p cross-section of the (real) differentiable vector bundle $L(s_{\div}^*TM, q_{\div}^*T\Gamma)$. Pulling it back along the diffeomorphism $a : \Gamma_s \times_t \Gamma \xrightarrow{\sim} \Gamma_{\div}$ given by $(g, k) \mapsto (gk, k)$, we obtain a C^p cross-section of the differentiable vector bundle $a^*L(s_{\div}^*TM, q_{\div}^*T\Gamma) \cong L(a^*s_{\div}^*TM, a^*q_{\div}^*T\Gamma) \cong L(pr_1^*s^*TM, pr_1^*T\Gamma) \cong pr_1^*L(s^*TM, T\Gamma)$ (where pr_1 denotes projection on the first factor from $\Gamma_s \times_t \Gamma$ onto Γ), which, applied to v , occurs in the expression (16a) as the integrand. The fundamental lemma about Haar integrals depending on parameters, Proposition B.12, immediately implies that $\hat{\eta}^H \in \Gamma^p(\Gamma; L(s^*TM, T\Gamma))$.

Let us proceed to check that $s_* \circ \hat{\eta}^H = id_{s^*TM}$.

$$\begin{aligned} (s_*)_g \circ \hat{\eta}_g^H &= T_g s \circ \int_{tk=sg} \delta^H(gk, k) d\nu_{sg}(k) \\ &= \int_{tk=sg} T_g s \circ (\eta_{gk}^H \div \eta_k^H) \circ (\lambda_k^H)^{-1} d\nu_{sg}(k) \end{aligned}$$

$$\begin{aligned}
&= \int_{tk=sg} (T_k t \circ \eta_k^H) \circ (\lambda_k^H)^{-1} d\nu_{sg}(k) \\
&= \int_{tk=sg} d\nu_{sg}(k) id_{T_{sg}M} = (id_{s^*TM})_g
\end{aligned}$$

To conclude, we must show that $\hat{\eta}_{1x}^H$ equals $T_x 1$ for all $x \in M$. We have $\delta^H(k, k) = (\eta_k^H \div \eta_k^H) \circ (\lambda_k^H)^{-1} = (1^{T\Gamma} \circ t^{T\Gamma} \circ \eta_k^H) \circ (\lambda_k^H)^{-1} = T_{tk} 1 \circ (T_k t \circ \eta_k^H) \circ (\lambda_k^H)^{-1} = T_{tk} 1$ and hence $\hat{\eta}_{1x}^H = \int d\nu_x(k) T_x 1 = T_x 1$. The proof is finished. \square

Just like any other groupoid connection on $\Gamma \rightrightarrows M$, the multiplicative average $\hat{\eta}^H$ will determine a pseudo-representation of Γ on TM , hereafter denoted by $\hat{\lambda}^H$, for which we have the following explicit formula in terms of λ^H .

$$\begin{aligned}
\hat{\lambda}_g^H &= T_g t \circ \hat{\eta}_g^H = \int_{tk=sg} T_g t \circ (\eta_{gk}^H \div \eta_k^H) \circ (\lambda_k^H)^{-1} d\nu_{sg}(k) \\
&= \int_{tk=sg} (T_{gk} t \circ \eta_{gk}^H) \circ (\lambda_k^H)^{-1} d\nu_{sg}(k) \\
&= \int_{tk=sg} \lambda_{gk}^H \circ (\lambda_k^H)^{-1} d\nu_{sg}(k)
\end{aligned} \tag{17}$$

Multiplicativity equations for the vertical component of a connection

In Section 2, when dealing with multiplicative connections on action groupoids, we exploited the natural splitting of the tangent bundle of an action groupoid into its vertical and its horizontal subbundle to the purpose of rewriting the multiplicativity condition in terms of the vertical component of a connection. This proved to be useful, then, from the point of view of computations. We intend to work out an analogous reformulation of the multiplicativity condition in the more general context of the present section. Even though in the case of an arbitrary proper Lie groupoid there are no global—let alone canonical—trivializations of the groupoid source mapping available, we can at least always find such trivializations at the infinitesimal level. Namely, on our groupoid $\Gamma \rightrightarrows M$ let us randomly fix some connection Φ , which we agree to call the “background” connection. This choice will give rise to a splitting of the tangent bundle of Γ into a vertical and a horizontal component relative to the tangent source mapping.

$$\begin{array}{ccc}
\pi^\Phi \stackrel{\text{def}}{\vdots} & T\Gamma \xrightarrow[\sim]{(\omega \circ \beta^\Phi, s_*)} t^*\mathfrak{g} \oplus s^*TM & (18) \\
& \swarrow s_* \quad \searrow pr_2 & \\
& s^*TM &
\end{array}$$

(In this diagram, as in Section 1, $\beta^\Phi = id_{T\Gamma} - \eta^\Phi \circ s_* : T\Gamma \rightarrow T^\uparrow\Gamma$ denotes the vertical projection associated to Φ ; ω denotes the Maurer–Cartan isomorphism between the s -vertical tangent bundle $T^\uparrow\Gamma \subset T\Gamma$ and the pullback of the algebroid bundle $\mathfrak{g} = 1^*T^\uparrow\Gamma$ along the target mapping.)

The splitting (18) leads to a corresponding decomposition of the tangent quotient operation (12). Namely, for each divisible pair of arrows $(g, h) \in \Gamma_\div$ there are two linear maps, hereafter denoted by $\dot{q}_{g,h}^\Phi$ and $\dot{s}_{g,h}^\Phi$, which go from the vector space $\mathfrak{g}_{tg} \oplus \mathfrak{g}_{th} \oplus$

$T_{sg=sh}M$ into respectively the vector space \mathfrak{g}_{tg} and the vector space $T_{th}M$ and which are characterized through the commutativity of the following diagram.

$$\begin{array}{ccc}
 T_g \Gamma & \xrightarrow[T_{gs} \times T_{hs}]{} & T_h \Gamma \xrightarrow[\sim]{\pi_g^\Phi \times \pi_h^\Phi} (\mathfrak{g}_{tg} \oplus T_{sg}M) \xrightarrow{pr_2} (\mathfrak{g}_{th} \oplus T_{sh}M) \\
 \parallel (1) & & \parallel \\
 T_{(g,h)}(\Gamma \div) & & \mathfrak{g}_{tg} \oplus \mathfrak{g}_{th} \oplus T_{sg=sh}M \\
 \downarrow T_{(g,h)}q_\div & & \downarrow (\dot{q}_{g,h}^\Phi, \dot{s}_{g,h}^\Phi) \\
 T_{gh^{-1}}\Gamma & \xrightarrow[\sim]{\pi_{gh^{-1}}^\Phi} & \mathfrak{g}_{tg=t(gh^{-1})} \oplus T_{th=s(gh^{-1})}M
 \end{array} \quad (19)$$

It is easily recognized that $\dot{s}_{g,h}^\Phi$ coincides with the composite linear map

$$\mathfrak{g}_{tg} \oplus \mathfrak{g}_{th} \oplus T_{sg=sh}M \xrightarrow{pr} \mathfrak{g}_{th} \oplus T_{sh}M \xrightarrow{(\pi_h^\Phi)^{-1}} T_h \Gamma \xrightarrow{T_h t} T_{th}M, \quad (20)$$

where pr of course denotes the projection $(X, Y, v) \mapsto (Y, v)$. The expression $\dot{s}_{g,h}^\Phi(X, Y, v)$ is then independent of g and X , so we shall abbreviate that expression into $\dot{s}_h^\Phi(Y, v)$. Let us introduce a bunch of related abbreviations, of which we will make repeated use.

$$\dot{q}_\uparrow^\Phi(g, h)(X, Y) := \dot{q}_{g,h}^\Phi(X, Y, 0) \quad \dot{s}_\uparrow^\Phi(h)Y := \dot{s}_h^\Phi(Y, 0) \quad (21a)$$

$$\dot{q}_\leftrightarrow^\Phi(g, h)v := \dot{q}_{g,h}^\Phi(0, 0, v) \quad \dot{s}_\leftrightarrow^\Phi(h)v := \dot{s}_h^\Phi(0, v) \quad (21b)$$

3.5. An arbitrary connection H on $\Gamma \rightrightarrows M$ will be entirely encoded into its *vertical component* $X^{H/\Phi}$ relative to the chosen background connection Φ . By definition, this is the vector bundle morphism of s^*TM into $t^*\mathfrak{g}$ given at each $g \in \Gamma$ by the linear map

$$X_g^{H/\Phi} \stackrel{\text{def}}{=} \omega_g \circ \beta_g^\Phi \circ \eta_g^H : T_{sg}M \rightarrow \mathfrak{g}_{tg}.$$

When the background connection Φ is fixed (as in our case), we may of course simply write ' X^H ' instead of ' $X^{H/\Phi}$ ', and so on. By the above definitions, we have $\pi_g(\eta_g^H v) = ((\omega_g \circ \beta_g)\eta_g^H v, (T_g s)\eta_g^H v) = (X_g^H v, v)$ for all tangent vectors $v \in T_{sg}M$. Since π_g is an invertible linear map, the condition (14) is satisfied if and only if the identity below holds for every divisible pair $(g, h) \in \Gamma \div$ for all $v \in T_{sg=sh}M$.

$$\begin{aligned}
 (X_{gh^{-1}}^H \lambda_h^H v, \lambda_h^H v) &= \pi_{gh^{-1}}(\eta_{gh^{-1}}^H \lambda_h^H v) \\
 &= \pi_{gh^{-1}}(\eta_g^H v \div \eta_h^H v) \\
 &= [\pi_{gh^{-1}} \circ T_{(g,h)}q_\div](\eta_g^H v, \eta_h^H v) \\
 &= (\dot{q}_{g,h}(X_g^H v, X_h^H v, v), \dot{s}_h(X_h^H v, v))
 \end{aligned}$$

Suppressing v in the last identity and making use of the shorthand (21), we obtain the following two equations.

$$X_{gh^{-1}}^H \circ \lambda_h^H = \dot{q}_\uparrow(g, h) \circ (X_g^H, X_h^H) + \dot{q}_\leftrightarrow(g, h) \quad (22a)$$

$$\lambda_h^H = \dot{s}_\uparrow(h) \circ X_h^H + \dot{s}_\leftrightarrow(h) \quad (22b)$$

Notice that the second of these equations is a tautology: indeed, by the above remark to the effect that $\dot{s}_h(Y, v)$ equals $(T_h t)\pi_h^{-1}(Y, v)$ [compare (20)], we have

$$\begin{aligned}\lambda_h^H - \dot{s}_h \circ (X_h^H, id) &= T_h t \circ \eta_h^H - T_h t \circ \pi_h^{-1} \circ (X_h^H, id) \\ &= T_h t \circ (\eta_h^H - \pi_h^H) = 0.\end{aligned}$$

It follows that for any connection H the condition (14) is equivalent to the following system of equations, which one obtains by substituting (22b) into (22a) and which only involve the vertical component X^H of H :

$$[\forall (g, h) \in \Gamma_{\div}] \quad \dot{q}_{\uparrow}(g, h) \circ (X_g^H, X_h^H) = X_{gh^{-1}}^H \circ (\dot{s}_{\downarrow}(h) \circ X_h^H + \dot{s}_{\leftrightarrow}(h)) - \dot{q}_{\leftrightarrow}(g, h). \quad (23)$$

3.6. In the above expressions (22) and (23), the horizontal terms $\dot{q}_{\leftrightarrow}(\dots)$ and $\dot{s}_{\leftrightarrow}(\dots)$ can be given a slightly more intuitive description, as follows. To begin with, we have $\dot{s}_h(0, v) = (T_h t)\pi_h^{-1}(0, v) = (T_h t)\eta_h v$ for all $v \in T_{sh}M$, whence

$$\dot{s}_{\leftrightarrow}(h) = \lambda_h. \quad (24)$$

Next, for every $v \in T_{sg=sh}M$ we have (at least when Φ is *non-degenerate*) $\dot{q}_{g,h}(0, 0, v) = (pr_1 \circ \pi_{gh^{-1}} \circ T_{(g,h)}q_{\div})(\pi_g^{-1}(0, v), \pi_h^{-1}(0, v)) = (\omega_{gh^{-1}} \circ \beta_{gh^{-1}} \circ T_{(g,h)}q_{\div})(\eta_g v, \eta_h v) = (\omega_{gh^{-1}} \circ \beta_{gh^{-1}})(\eta_g v \div \eta_h v) = [\omega_{gh^{-1}} \circ \beta_{gh^{-1}} \circ \delta(g, h)]\lambda_h v$. Thus, letting $\Delta^{\Phi}(g, h) := \omega_{gh^{-1}} \circ \beta_{gh^{-1}}^{\Phi} \circ \delta^{\Phi}(g, h) (: T_{th}M \rightarrow \mathfrak{g}_{t(gh^{-1})=tg})$ denote the vertical component of the difference cocycle,

$$\dot{q}_{\leftrightarrow}(g, h) = \Delta(g, h) \circ \lambda_h. \quad (25)$$

Substituting the last two expressions into (23), we reach the following conclusion: *For any choice of a non-degenerate background connection Φ on $\Gamma \rightrightarrows M$, a unital connection H on $\Gamma \rightrightarrows M$ is multiplicative if, and only if, its vertical component $X^{H/\Phi}$ relative to Φ satisfies the following equation for every divisible pair $(g, h) \in \Gamma_{\div}$:*

$$\text{(Multiplicativity equation)} \quad \boxed{\dot{q}_{\uparrow}^{\Phi}(g, h) \circ (X_g^{H/\Phi}, X_h^{H/\Phi}) = X_{gh^{-1}}^{H/\Phi} \circ \lambda_h^H - \Delta^{\Phi}(g, h) \circ \lambda_h^{\Phi}} \quad (26a)$$

$$\text{where} \quad \lambda_h^H = \lambda_h^{\Phi} + \dot{s}_{\uparrow}^{\Phi}(h) \circ X_h^{H/\Phi}. \quad (26b)$$

Cocycle equations for the background connection

Let $g, h, k \in \Gamma$ satisfy $sg = sh = sk$. Then

$$q_{\div}(g, k) = gk^{-1} = gh^{-1}hk^{-1} = gh^{-1}(kh^{-1})^{-1} = q_{\div}(q_{\div}(g, h), q_{\div}(k, h)).$$

If for each pair of indices $i, j \in \{1, 2, 3\}$ with $i \neq j$ we let q_{ij} denote the mapping from $\Gamma_{s \times_s} \Gamma_{s \times_s} \Gamma$ into Γ given by $(g_1, g_2, g_3) \mapsto q_{\div}(g_i, g_j)$, we can rephrase the last identity more succinctly as

$$q_{13} = q_{\div} \circ (q_{12}, q_{32}).$$

Differentiating this identity at $(g, h, k) \in \Gamma_{s \times_s} \Gamma_{s \times_s} \Gamma$ and taking into account the obvious relations $T_{(g_1, g_2, g_3)}q_{ij} = T_{(g_i, g_j)}q_{\div} \circ pr_{ij}$ where pr_{ij} denotes the map of $T_{g_1} \Gamma_{T_{g_1} s \times T_{g_2} s} T_{g_2} \Gamma_{T_{g_2} s \times T_{g_3} s} T_{g_3} \Gamma$ onto $T_{g_i} \Gamma_{T_{g_i} s \times T_{g_j} s} T_{g_j} \Gamma$ given by $(w_1, w_2, w_3) \mapsto (w_i, w_j)$, we obtain

$$T_{(g, k)}q_{\div} \circ pr_{13} = T_{(gh^{-1}, kh^{-1})}q_{\div} \circ (T_{(g, h)}q_{\div} \circ pr_{12}, T_{(k, h)}q_{\div} \circ pr_{32}).$$

Composing to the left with the invertible linear map $\pi_{gk^{-1}=gh^{-1}(kh^{-1})^{-1}}$ and to the right with the linear map $(\eta_g, \eta_h, \eta_k) : T_{sg=sh=sk}M \rightarrow T_g\Gamma \times_{T_g s \times T_h s} T_h\Gamma \times_{T_h s \times T_k s} T_k\Gamma$, and making repeated use of the commutativity of the diagram (19) and of the relation $\pi \circ \eta = (0, id_{s^*TM})$, we obtain the following pair of equations, in which $v \in T_{sg=sh=sk}M$.

$$\begin{aligned}\dot{q}_{g,k}(0, 0, v) &= \dot{q}_{gh^{-1}, kh^{-1}}(\dot{q}_{g,h}(0, 0, v), \dot{q}_{k,h}(0, 0, v), \dot{s}_h(0, v)) \\ \dot{s}_k(0, v) &= \dot{s}_{kh^{-1}}(\dot{q}_{k,h}(0, 0, v), \dot{s}_h(0, v))\end{aligned}$$

Next, recalling our notational conventions (21) and the identities (24) and (25) (as in 3.6, we shall henceforth assume that the given background connection Φ is *non-degenerate*), we rewrite these equations as follows.

$$\begin{aligned}\Delta(g, k)\lambda_k v &= \dot{q}_{\uparrow}(gh^{-1}, kh^{-1})(\dot{q}_{\leftrightarrow}(g, h)v, \dot{q}_{\leftrightarrow}(k, h)v) \\ &\quad + \dot{q}_{\leftrightarrow}(gh^{-1}, kh^{-1})\dot{s}_{\leftrightarrow}(h)v \\ &= [\dot{q}_{\uparrow}(gh^{-1}, kh^{-1}) \circ (\Delta(g, h), \Delta(k, h))]\lambda_h v \\ &\quad + [\Delta(gh^{-1}, kh^{-1}) \circ \lambda_{kh^{-1}}]\lambda_h v \\ \lambda_k v &= \dot{s}_{\uparrow}(kh^{-1})\dot{q}_{\leftrightarrow}(k, h)v + \dot{s}_{\leftrightarrow}(kh^{-1})\dot{s}_{\leftrightarrow}(h)v \\ &= [\dot{s}_{\uparrow}(kh^{-1}) \circ \Delta(k, h)]\lambda_h v + \lambda_{kh^{-1}}\lambda_h v\end{aligned}$$

After suppressing v from these equations and setting $kh^{-1} = h'$ in the second of them, we are left with the following tautological expressions, which we call “cocycle equations”.

$$\begin{aligned}\dot{q}_{\uparrow}(gh^{-1}, kh^{-1}) \circ (\Delta(g, h), \Delta(k, h)) \circ \lambda_h &= \Delta(g, k) \circ \lambda_k \\ &\quad - \Delta(gh^{-1}, kh^{-1}) \circ \lambda_{kh^{-1}} \circ \lambda_h\end{aligned}\tag{27a}$$

$$\lambda_{h'h} - \lambda_{h'} \circ \lambda_h = \dot{s}_{\uparrow}(h') \circ \Delta(h'h, h) \circ \lambda_h\tag{27b}$$

Proposition 3.7. *Suppose that Φ is an effective (hence non-degenerate) connection on a proper Lie groupoid $\Gamma \rightrightarrows M$. Then, the multiplicative average $\hat{\Phi}$ of Φ (computed with respect to any normalized left Haar system ν on $\Gamma \rightrightarrows M$) is a multiplicative connection.*

Proof. We must check the validity of the multiplicativity equation (26a) for the (in view of Lemma 3.4, unital) connection $H = \hat{\Phi}$, relative to the (non-degenerate) background connection Φ . It will not be a bad idea to abridge ‘ $\lambda^{\hat{\Phi}}$ ’ into ‘ $\hat{\lambda}$ ’ and ‘ $X^{\hat{\Phi}/\Phi}$ ’ into ‘ \hat{X} ’. We shall also systematically suppress ‘ Φ ’-superscripts and thus, for instance, simply write ‘ $\Delta(g, h)$ ’ in place of ‘ $\Delta^{\Phi}(g, h)$ ’ and ‘ λ_g ’ in place of ‘ λ_g^{Φ} ’.

As a side remark, we observe that when $H = \hat{\Phi}$ the tautology (26b)—which, we stress, is merely a consequence of H being a connection—can alternatively be deduced from the cocycle equation (27b). Indeed, referring back to the formula (17), we have

$$\begin{aligned}\hat{\lambda}_{h'} - \lambda_{h'} &= \int_{th=sh'} \lambda_{h'h}\lambda_h^{-1} - \lambda_{h'} d\nu_{sh'}(h) \\ &= \int_{th=sh'} \dot{s}_{\uparrow}(h') \circ \Delta(h'h, h) d\nu_{sh'}(h) \quad [\text{by (27b)}] \\ &= \dot{s}_{\uparrow}(h') \circ \hat{X}_{h'}.\end{aligned}\tag{28}$$

We proceed to check the validity of the multiplicativity equation for $\hat{\Phi}$.

$$\dot{q}_{\uparrow}(g, h) \circ (\hat{X}_g, \hat{X}_h) = \dot{q}_{\uparrow}(g, h) \circ \int_{tk=sg=sh=x} (\Delta(gk, k), \Delta(hk, k)) d\nu_x(k)$$

$$\begin{aligned}
 &= \int_{tk=x} \dot{q}_{\uparrow}(g, h) \circ (\Delta(gk, k), \Delta(hk, k)) d\nu_x(k) \\
 [\text{By (27a):}] &= \int [\Delta(gk, hk) \circ \lambda_{hk} \circ \lambda_k^{-1} - \Delta(g, h) \circ \lambda_h] d\nu_x(k) \\
 [\text{By (27b):}] &= \int \Delta(gk, hk) \circ [\dot{s}_{\uparrow}(h) \circ \Delta(hk, k) + \lambda_h] d\nu_x(k) \\
 &\quad - \int d\nu_x(k) \Delta(g, h) \circ \lambda_h \\
 [\text{By (28):}] &= \int \Delta(gk, hk) \circ \dot{s}_{\uparrow}(h) \circ \Delta(hk, k) d\nu_x(k) \\
 &\quad + \int \Delta(gk, hk) \circ [\hat{\lambda}_h - \dot{s}_{\uparrow}(h) \circ \hat{X}_h] d\nu_{x=sh}(k) - \Delta(g, h) \circ \lambda_h \\
 &= \int \Delta(gk, hk) \circ \dot{s}_{\uparrow}(h) \circ [\Delta(hk, k) - \hat{X}_h] d\nu_x(k) \\
 &\quad + \int_{tk'=th} \Delta(gh^{-1}k', k') \circ \hat{\lambda}_h d\nu_{th}(k') - \Delta(g, h) \circ \lambda_h \\
 &= \iint \Delta(gk, hk) \circ \dot{s}_{\uparrow}(h) \circ [\Delta(hk, k) - \Delta(hk', k')] d\nu_x(k) d\nu_x(k') \\
 &\quad + \hat{X}_{gh^{-1}} \circ \hat{\lambda}_h - \Delta(g, h) \circ \lambda_h
 \end{aligned}$$

Thus far we have not used the assumption that Φ was effective. Now, if that is the case then by (27b) the double integral term must vanish. \square

A connection Φ on a proper Lie group bundle $\Gamma \xrightarrow{s=l} M$ is always effective since for any such groupoid the target mapping equals the source mapping and therefore $\lambda_g^\Phi = id_{T_{sg}M}$ for all $g \in \Gamma$. Thus the preceding proposition has the following immediate consequence:

Corollary 3.8. *Any proper Lie group bundle admits multiplicative connections.*

3.9 (Concluding remarks). The equations (27) look like a reworking of the equations (5) and (7) that appear in the statement of [1, Proposition 2.15], which underlie the construction of the adjoint representation of a Lie groupoid as a representation up to homotopy. We decided to call our equations “cocycle equations” simply by analogy with the above-mentioned reference. In fact, we have not even attempted giving a really convincing justification for our choice of terminology, partly because this issue does not seem so relevant for the purposes of the present paper. It would probably be instructive to look for a more conceptual definition of the averaging operator (16) in the framework of Lie groupoid cohomology with up-to-homotopy coefficients, hopefully in this way shedding more light on the role of the non-degeneracy condition appearing in our definition and of the “longitudinal” obstruction arising from the fact that the effect of a connection may in general fail to be a representation. In particular, we believe there should be a way of relating our Proposition 3.7 with the cohomology vanishing theorem of [1]. Evidence in this direction also seems to be provided by the deformation argument used by Weinstein in the proof of [30, Theorem 7.1], which may be turned into a proof of Corollary 3.8 above. Anyway, the approach we adopt here seems to us preferable at least from the point of view of a self-contained exposition. In any case, we do not really know how to adapt the arguments of [30] to a situation more general than that considered in Corollary 3.8.

4. Basic recursive estimates

In the preceding section, we saw that the process that to each non-degenerate connection Φ on a proper Lie groupoid $\Gamma \rightrightarrows M$ assigns the corresponding multiplicative average $\hat{\Phi}$ (taken with respect to some fixed normalized left Haar system on $\Gamma \rightrightarrows M$) results for each order of differentiability $p = 0, 1, 2, \dots, \infty$ in an *averaging operator*

$$\text{Conn}_*^p(\Gamma) \longrightarrow \text{Conn}_u^p(\Gamma), \quad \Phi \mapsto \hat{\Phi}.$$

We have shown that this operator carries effective connections into multiplicative connections. It is immediate to see that every multiplicative connection belongs to the fixed point set of this operator. On the basis of these considerations, for a generic non-degenerate connection Φ it seems interesting to investigate the sequence of connections $\hat{\Phi}, \widehat{\hat{\Phi}}, \dots$ which one obtains by repeatedly averaging Φ . Of course, there is in the first place the question of whether this sequence is at all defined. Provided it is, one can study its convergence. In view of the integral formula (17) and of Proposition 3.7, it is reasonable to expect the behavior in the limit of the iterated averages of Φ to depend essentially on the behavior in the limit of their effects. These are known to coincide with the pseudo-representations of $\Gamma \rightrightarrows M$ on TM obtained from the pseudo-representation λ^Φ by recursive application of the formula (17). At this point, there is no particular reason for restricting one's analysis only to pseudo-representations arising from connections.

As in the previous section, we shall be assigned a proper Lie groupoid $\Gamma \rightrightarrows M$ and, on it, a normalized left Haar system ν . In addition, we shall be given a \mathbb{K} -linear ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) differentiable vector bundle E over the base M of Γ . These data will be regarded as fixed for the rest of the section.

4.1. For each order of differentiability $p = 0, 1, 2, \dots, \infty$ let

$$\text{Psr}_*^p(\Gamma; E) \subset \text{Psr}^p(\Gamma; E)$$

(by analogy with the notations introduced in Section 3) denote the set of all invertible C^p pseudo-representations of Γ on E . If for any pseudo-representation λ of Γ on TM which for some non-degenerate groupoid connection Φ is of the form $\lambda = \lambda^\Phi$ we set

$$\Delta^\lambda(h'h, h) := \hat{s}_\dagger(h') \circ \Delta(h'h, h)$$

in the tautological expression (27b) and then take $h' = gh^{-1}$, we obtain the identity

$$\Delta^\lambda(g, h) = \lambda_g \circ \lambda_h^{-1} - \lambda_{gh^{-1}}. \quad (29)$$

Regarding this as the definition of Δ^λ when λ is an arbitrary invertible pseudo-representation of Γ on E , we obtain a global, vector bundle cross-section

$$\Delta^\lambda \in \Gamma(\Gamma_\div; L(s_\div^* E, t_\div^* E)).$$

This cross-section will be of class C^p whenever the given pseudo-representation λ is of class C^p . Motivated by the identity (17), which expresses the effect of the multiplicative

average of a connection in terms of the effect of the connection itself, for any given invertible C^p pseudo-representation $\lambda \in \text{Psr}_*^p(\Gamma; E)$ and for every arrow $g \in \Gamma$ we set

$$\hat{\lambda}(g) \stackrel{\text{def}}{=} \int_{tk=sg} \lambda(gk) \circ \lambda(k)^{-1} dv_{sg}(k). \quad (30)$$

By the same argument as in the proof of Lemma 3.4, it follows from the fundamental lemma on Haar integrals depending on parameters (Proposition B.12) that $\hat{\lambda}$ belongs to $\text{Psr}_*^p(\Gamma; E)$; otherwise stated, $\hat{\lambda}$ is a unital C^p pseudo-representation of Γ on E .

We have the following two fundamental equations.

$$\begin{aligned} \hat{\lambda}(g'g) - \hat{\lambda}(g') \circ \hat{\lambda}(g) &= \int_{tk=sg} \Delta^\lambda(g'gk, gk) \circ \Delta^\lambda(gk, k) dv_{sg}(k) \\ &\quad - \iint_{\substack{th=sg \\ tk=sg}} \Delta^\lambda(g'gh, gh) \circ \Delta^\lambda(gk, k) dv_{sg}(h) dv_{sg}(k) \end{aligned} \quad (31a)$$

$$\hat{\lambda}(g) = \lambda(g) + \int_{tk=sg} \Delta^\lambda(gk, k) dv_{sg}(k) \quad (31b)$$

Proof. The second equation is an immediate consequence of our Haar system's being normalized: $\hat{\lambda}_g = \int \lambda_{gk} \lambda_k^{-1} dv_{sg}(k) = \int \lambda_g dv_{sg}(k) + \int (\lambda_{gk} \lambda_k^{-1} - \lambda_g) dv_{sg}(k) = \lambda_g + \int \Delta^\lambda(gk, k) dv_{sg}(k)$. As to the first equation, we use both left invariance and normality of the Haar integral:

$$\begin{aligned} \hat{\lambda}_{g'g} - \hat{\lambda}_{g'} \hat{\lambda}_g &= \int \lambda_{g'gk} \lambda_k^{-1} dv_{sg}(k) - \left(\int \lambda_{g'k'} \lambda_{k'}^{-1} dv_{sg'=tg}(k') \right) \circ \left(\int \lambda_{gk} \lambda_k^{-1} dv_{sg}(k) \right) \\ [\text{setting } k' = gh:] &= \int \dots - \left(\int \lambda_{g'gh} \lambda_{gh}^{-1} dv_{sg}(h) \right) \circ \left(\int \dots \right) \\ &= \int \lambda_{g'gk} \lambda_k^{-1} dv_{sg}(k) - \int \lambda_{g'gk} \lambda_{gk}^{-1} \lambda_g dv_{sg}(k) \\ &\quad - \int \lambda_{g'} \lambda_{gk} \lambda_k^{-1} dv_{sg}(k) + \lambda_{g'} \lambda_g \\ &\quad - \iint \lambda_{g'gh} \lambda_{gh}^{-1} \lambda_{gk} \lambda_k^{-1} dv_{sg}(h) dv_{sg}(k) + \int \lambda_{g'gh} \lambda_{gh}^{-1} \lambda_g dv_{sg}(h) \\ &\quad + \int \lambda_{g'} \lambda_{gk} \lambda_k^{-1} dv_{sg}(k) - \lambda_{g'} \lambda_g \\ &= \int (\lambda_{g'gk} \lambda_{gk}^{-1} - \lambda_{g'}) \circ (\lambda_{gk} \lambda_k^{-1} - \lambda_g) dv_{sg}(k) \\ &\quad - \iint (\lambda_{g'gh} \lambda_{gh}^{-1} - \lambda_{g'}) \circ (\lambda_{gk} \lambda_k^{-1} - \lambda_g) dv_{sg}(h) dv_{sg}(k), \end{aligned}$$

which is the desired relation. \square

4.2. Let us endow the given vector bundle E with some metric ϕ of class C^∞ (Riemannian or Hermitian, depending on whether E is real or complex). We shall keep ϕ fixed throughout the sequel. For each pair of base points $x, y \in M$ we obtain a norm $\| \cdot \|_{x,y}$ on $L(E_x, E_y)$ upon setting

$$\| \lambda \|_{x,y} = \sup_{|v|_x=1} | \lambda v |_y$$

for every linear map $\lambda : E_x \rightarrow E_y$. [Here of course $| \cdot |_x$ denotes the norm on E_x given by $|v|_x = \sqrt{\phi_x(v, v)}$.] Observe that the norms $\| \cdot \|_{x,y}$ satisfy the inequalities

$$\| \mu \circ \lambda \|_{x,z} \leq \| \mu \|_{y,z} \| \lambda \|_{x,y}. \quad (32)$$

It follows in particular that $\text{End}(E_x)$ is a Banach algebra under the norm $\| \cdot \|_{x,x}$.

Lemma 4.3. *Let A be a Banach algebra with unit element e . Let a real constant $0 \leq c < 1$ be assigned. Then, for every element v of A such that $|v| \leq c$, the element $e - v$ is invertible and*

$$|(e - v)^{-1} - e| \leq c(1 - c)^{-1}.$$

Proof. Since $|v| < 1$, the element $e - v$ is invertible with inverse

$$(e - v)^{-1} = e + v + v^2 + v^3 + \dots$$

so that $|(e - v)^{-1} - e| \leq |v| + |v|^2 + |v|^3 + \dots = |v|(1 - |v|)^{-1} \leq c(1 - c)^{-1}$. \square

4.4. For every continuous pseudo-representation λ of Γ on E let us set

$$b_\phi(\lambda) \stackrel{\text{def}}{=} \sup_{g \in \Gamma} \|\lambda(g)\|_{sg, tg} \quad \text{and} \quad (33a)$$

$$c_\phi(\lambda) \stackrel{\text{def}}{=} \sup_{(g', g) \in \Gamma_s \times_t \Gamma} \|\lambda(g'g) - \lambda(g') \circ \lambda(g)\|_{sg, tg'}. \quad (33b)$$

These quantities may be infinite, of course. However if $b_\phi(\lambda) < \infty$ then $c_\phi(\lambda) < \infty$. Since ϕ is fixed, we shall simply write ‘ $b(\lambda)$ ’ and ‘ $c(\lambda)$ ’ throughout the sequel.

4.5 (Remark). *Let $\lambda \in \text{Psr}_u^0(\Gamma; E)$ be a unital, continuous, pseudo-representation of Γ on E . Suppose that $c(\lambda) < 1$. Then λ must be invertible. [Proof. Our assumptions entail that $1 > \|id - \lambda_{g^{-1}} \circ \lambda_g\|_{sg, sg}$ for all g . Since $\text{End}(E_{sg})$ equipped with the norm $\|\cdot\|_{sg, sg}$ is a unital Banach algebra, it follows that $\lambda_{g^{-1}} \circ \lambda_g$ must be an invertible element of $\text{End}(E_{sg})$ and therefore that λ_g must be a left invertible (hence, injective) linear map. Similarly, we see that λ_g must be right invertible (hence surjective).]*

4.6. *One has the two estimates below, for any unital, continuous, pseudo-representation $\lambda \in \text{Psr}_u^0(\Gamma; E)$ which satisfies the condition $c(\lambda) < 1$.*

$$[\forall g \in \Gamma] \quad \|\lambda(g)^{-1}\|_{tg, sg} \leq \frac{b(\lambda)}{1 - c(\lambda)} \quad (34a)$$

$$[\forall (g, h) \in \Gamma_{\div}] \quad \|\Delta^\lambda(g, h)\|_{th, tg} \leq c(\lambda) \frac{b(\lambda)}{1 - c(\lambda)} \quad (34b)$$

(*Comment.* One can make sense of the last inequality so that this remains correct even when $b(\lambda) = \infty$ if one makes the convention that $0 \cdot \infty = 0$.) [Proof. To show the inequality (34a), we apply Lemma 4.3 to the Banach algebra $A = \text{End}(E_{sg})$ (of course equipped with the norm $\|\cdot\|_{sg, sg}$), to the number $c = c(\lambda) < 1$, and to the algebra element $v = id - \lambda_{g^{-1}} \circ \lambda_g$. We get (omitting norm subscripts)

$$\|\lambda_g^{-1} \circ \lambda_{g^{-1}} - id\| = \|[id - (id - \lambda_{g^{-1}} \circ \lambda_g)]^{-1} - id\| \leq c(1 - c)^{-1}.$$

Using the inequalities (32) we obtain

$$\|\lambda_g^{-1} - \lambda_{g^{-1}}\| = \|(\lambda_g^{-1} \circ \lambda_{g^{-1}} - id) \circ \lambda_{g^{-1}}\| \leq c(1 - c)^{-1} \|\lambda_{g^{-1}}\|,$$

whence, finally,

$$\|\lambda_g^{-1}\| \leq \|\lambda_{g^{-1}}\| + \|\lambda_g^{-1} - \lambda_{g^{-1}}\| \leq \left(1 + \frac{c}{1 - c}\right) \|\lambda_{g^{-1}}\| = \frac{\|\lambda_{g^{-1}}\|}{1 - c}.$$

As to (34b), this is an immediate consequence of (34a):

$$\|\lambda_g \lambda_h^{-1} - \lambda_{gh^{-1}}\| = \|(\lambda_g - \lambda_{gh^{-1}} \lambda_h) \circ \lambda_h^{-1}\| \leq c(\lambda) \|\lambda_h^{-1}\|.$$

4.7. For each unital, continuous, pseudo-representation $\lambda \in \text{Ps}_u^0(\Gamma; E)$ which satisfies the condition $c(\lambda) < 1$, the pseudo-representation $\hat{\lambda} \in \text{Ps}_u^0(\Gamma; E)$ defined by (30) conforms with the following numerical constraints.

$$[\forall g \in \Gamma] \quad \|\hat{\lambda}(g)\|_{sg, tg} \leq \frac{b(\lambda)}{1 - c(\lambda)} \quad (35a)$$

$$[\forall (g', g) \in \Gamma \times_s \Gamma] \quad \|\hat{\lambda}(g'g) - \hat{\lambda}(g') \circ \hat{\lambda}(g)\|_{sg, tg'} \leq 2c(\lambda)^2 \frac{b(\lambda)^2}{[1 - c(\lambda)]^2} \quad (35b)$$

[*Proof.* The two estimates are an immediate consequence of the preceding inequalities (34) and of the identities (31). Indeed, the Haar system ν involved in (31) is normalized, so one can estimate each one of the integrals that appear in (31) simply by the sup norm of its integrand.]

Lemma 4.8. Let $\{b_0, b_1, \dots, b_l\}$ and $\{c_0, c_1, \dots, c_l\}$ be two finite sequences of non-negative real numbers, of length, say, $l + 1 \geq 2$. Suppose that for every index i between 0 and $l - 1$ the following implication is true.

$$c_i < 1 \Rightarrow \begin{cases} b_{i+1} \leq \frac{b_i}{1 - c_i} & \text{and} \\ c_{i+1} \leq 2c_i^2 \left[\frac{b_i}{1 - c_i} \right]^2 \end{cases} \quad (36)$$

Also suppose that $b_0 \geq 1$ and that $\varepsilon = 6b_0^2 c_0 \leq \frac{2}{3}$. Then, the following inequalities must hold for every index $i = 0, 1, \dots, l$.

$$c_i \leq \frac{\varepsilon^{2^i}}{6b_0^2} \quad (37a)$$

$$\frac{b_i}{1 - c_i} \leq \sqrt{3}b_0 \quad (37b)$$

Proof. We claim that if the inequality (37a) holds for every index i between zero and a certain non-negative integer $n \leq l$ then the inequality (37b) must hold upon taking $i = n$. Indeed, let the condition $c_i \leq \varepsilon^{2^i} / (6b_0^2)$ be satisfied for all indices $i = 0, \dots, n$, and suppose—but only provisionally—that $n > 0$. Under such assumptions, for all indices $i \leq n - 1$ we must have $c_i < 1$ (because by hypothesis $b_0 \geq 1$ and $\varepsilon < 1$) and therefore $b_{i+1} \leq b_i / (1 - c_i)$ [by (36)]. Combining recursively all these inequalities as i runs from zero to $n - 1$, we conclude that $b_n \leq b_0 / (1 - c_0) \cdots (1 - c_{n-1})$. Thus

$$b_n / (1 - c_n) \leq b_0 / (1 - c_0) \cdots (1 - c_n). \quad (38)$$

This inequality is obviously also true when $n = 0$. We proceed to study the quantity

$$1 / \prod_{i=0}^n (1 - c_i) = [\exp \log \prod_{i=0}^n (1 - c_i)]^{-1} = \exp(-\sum_{i=0}^n \log(1 - c_i)).$$

For every real number x such that $|x| < 1$ we have $-|x| + |\log(1+x)| \leq |x - \log(1+x)| = \left| \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \dots \right| \leq \frac{|x|^2}{2} + \frac{|x|^3}{2} + \frac{|x|^4}{2} + \dots = \frac{|x|^2}{2} \frac{1}{1-|x|}$. This quantity is $\leq |x|^2$ whenever $|x| \leq 1/2$. Hence, setting $x = -t$, we see that

$$-\log(1-t) \leq t + t^2 \quad \forall t \in [0, \frac{1}{2}]. \quad (39)$$

Since $2^i \geq 2i$ for every integer $i \geq 0$, and since by hypothesis $b_0 \geq 1$ and $\varepsilon \leq 2/3 < 1$, for all indices $0 \leq i \leq n$ it must be true that $c_i \leq \varepsilon^{2^i}/(6b_0^2) \leq \varepsilon^{2^i}/6$. In particular, for all indices $0 \leq i \leq n$ it must be true that $c_i < 1/2$, whence, by (39),

$$\begin{aligned} \exp\left(\sum_{i=0}^n -\log(1-c_i)\right) &\leq \exp\left(\sum_{i=0}^n c_i + c_i^2\right) \\ &\leq \exp\left(\frac{1}{6} \sum_{i=0}^n \varepsilon^{2^i}\right) \exp\left(\frac{1}{6^2} \sum_{i=0}^n \varepsilon^{4^i}\right) \\ &\leq \exp\left(\frac{1}{6} \frac{1}{1-\varepsilon^2}\right) \exp\left(\frac{1}{6^2} \frac{1}{1-\varepsilon^4}\right) \\ &\leq \exp\left(\frac{1}{6} \frac{9}{5} + \frac{1}{6^2} \frac{81}{65}\right) \leq \exp\left(\frac{1}{2} \left[\frac{3}{5} + \frac{1}{5} \frac{1}{2} \frac{9}{13}\right]\right) \\ &\leq \exp(1/2) \leq \sqrt{3}. \end{aligned}$$

Combining the above with (38), we obtain the desired inequality $b_n/(1-c_n) \leq \sqrt{3}b_0$.

To finish the proof of the lemma, we will show that the inequality (37a) holds for every index i between zero and n by reasoning inductively on n . By hypothesis, the claim is true when $n = 0$. Assume that the claim holds for a certain value of $n \geq 0$, $\leq l-1$. Then, by the above, $b_n/(1-c_n) \leq \sqrt{3}b_0$, whence, by (36),

$$c_{n+1} \leq 2c_n^2 \left[\frac{b_n}{1-c_n} \right]^2 \leq 2 \frac{(\varepsilon^{2^n})^2}{(6b_0^2)^2} 3b_0^2 = \frac{\varepsilon^{2^{n+1}}}{6b_0^2}. \quad \square$$

4.9 (Concluding remarks). Fast convergence of recursive approximation processes is a familiar phenomenon in diverse mathematical contexts. It generally applies to the construction of exact solutions to problems which, a priori, are only known to admit approximate solutions (Newton's method for finding zeros of mappings [15, p. 139] being a basic example). In the theory of topological groups, it can be used to prove the existence of a homomorphism near any given "almost homomorphism" between two compact Lie groups [10], or the existence, for any compact group, of a representation by bounded Hilbert space operators near any given "approximate representation" [7]. In the more general context of groupoids, the use of a recursive averaging process was first proposed by Weinstein [29] as a possible technique for proving a conjecture of his about the local linearizability of proper Lie groupoids around fixed points; Weinstein's suggestion was successfully implemented by Zung [31]. Even though the underlying general principles are of course the same, it is to us not entirely clear how the specific computations in the present section relate to those in the above-mentioned references (in particular, our arguments appear to be simpler than those of de la Harpe and Karoubi [7]). Similar considerations can be made in comparison with [31] regarding the computations in the next two sections.

5. Fast convergence theorem I (pseudo-representations)

Let $\lambda \in \text{Psr}^0(\Gamma; E)$ be an arbitrary continuous pseudo-representation of a proper Lie groupoid $\Gamma \rightrightarrows M$ on a (real or complex) differentiable vector bundle E over the base M of Γ . For any (non-empty) open subset V of M and for any (Riemannian or Hermitian) metric ψ of class C^∞ on $E|V$, we can apply the definitions (33) to the pseudo-representation $\lambda|V \in \text{Psr}^0(\Gamma|V; E|V)$ that λ induces (upon restriction) on the open subgroupoid $\Gamma|V \rightrightarrows V$ of $\Gamma \rightrightarrows M$, thus obtaining a pair of (possibly infinite) quantities, which by analogy with 4.4 we shall indicate by $b_{V,\psi}(\lambda)$ and $c_{V,\psi}(\lambda)$.

Definition 5.1. We shall say that a unital continuous pseudo-representation $\lambda \in \text{Psr}_u^0(\Gamma; E)$ is *nearly multiplicative* or a *near representation* if each base point of Γ possesses an invariant open neighborhood $V = \Gamma V$ with the property that the inequality below holds for some choice of C^∞ metrics ψ on $E|V$.

$$c_{V,\psi}(\lambda) \leq \frac{1}{9} b_{V,\psi}(\lambda)^{-2} \quad (40)$$

[For $b_{V,\psi}(\lambda) = \infty$ this condition simply reads $c_{V,\psi}(\lambda) = 0$.] We shall say that a unital groupoid connection Φ on Γ is *nearly multiplicative* (even though it would be more appropriate to call it *nearly effective*) if the associated pseudo-representation λ^Φ of Γ on the tangent bundle of M , that is, the effect of Φ , is a near representation.

Our first comment is that a near representation $\lambda \in \text{Psr}_u^0(\Gamma; E)$ is always invertible. This is an obvious corollary to the remark 4.5. Indeed, by the condition (40), locally around each point in M we must have $c_{V,\psi}(\lambda) \leq \frac{1}{9} < 1$, because $b_{V,\psi}(\lambda) \geq 1$ in consequence of the unitality of λ .

In view of invertibility, for any near representation $\lambda \in \text{Psr}_u^0(\Gamma; E)$ it makes sense to consider the pseudo-representation $\hat{\lambda} \in \text{Psr}_u^0(\Gamma; E)$ that one obtains by averaging λ with respect to a given normalized left Haar system by means of the formula (30). We contend that the pseudo-representation obtained in this way is itself nearly multiplicative. Indeed, let V and ψ be as in the above definition and in particular satisfy (40). Let us abbreviate ' $b_{V,\psi}(\lambda)$ ', ' $c_{V,\psi}(\lambda)$ ', ' $b_{V,\psi}(\hat{\lambda})$ ' and ' $c_{V,\psi}(\hat{\lambda})$ ' respectively into ' b ', ' c ', ' \hat{b} ' and ' \hat{c} '. By the unitality of λ , we have $b \geq 1$ and hence $c \leq \frac{1}{9} < 1$. By the estimate (35a), we have $\hat{b} \leq b/(1-c) \leq \frac{9}{8}b$ and, consequently, $\hat{b}^{-2} \geq (\frac{8}{9})^2 b^{-2} \geq \frac{1}{2} b^{-2}$. Furthermore, by the estimate (35b) and by (40), we have $\hat{c} \leq 2c^2 b^2 / (1-c)^2 \leq 2(1/9b^2)^2 b^2 / (1 - \frac{1}{9})^2 = 2(\frac{1}{9})^2 (\frac{9}{8})^2 b^{-2} \leq \frac{1}{9} \frac{1}{2} b^{-2}$. Thus $\hat{c} \leq \frac{1}{9} \hat{b}^{-2}$.

From these remarks we conclude that any near representation $\lambda \in \text{Psr}_u^p(\Gamma; E)$ gives rise to a sequence $\{\hat{\lambda}^{(i)}\}_{i=0}^\infty$ of *averaging iterates* $\hat{\lambda}^{(i)} \in \text{Psr}_u^p(\Gamma; E)$ constructed recursively by setting $\hat{\lambda}^{(0)} := \lambda$ and $\hat{\lambda}^{(i+1)} := (\hat{\lambda}^{(i)})^\wedge$. (This construction depends of course on the preliminary choice of a normalized left Haar system on Γ .) The formula (17) says that λ^Φ equals $(\lambda^\Phi)^\wedge$ for any non-degenerate connection Φ of class C^0 on Γ , so we also conclude that any nearly multiplicative connection $\Phi \in \text{Conn}_u^p(\Gamma)$ gives rise, by recursive averaging, to a sequence of nearly multiplicative connections $\hat{\Phi}^{(i)} \in \text{Conn}_u^p(\Gamma)$ ($i = 0, 1, 2, \dots$).

Theorem 5.2. *Let $\Gamma \rightrightarrows M$ be an arbitrary proper Lie groupoid. Let $\lambda \in \text{Psr}_u^p(\Gamma; E)$ be a unital pseudo-representation of class C^p ($p = 0, 1, 2, \dots, \infty$) of Γ on some differentiable vector bundle E over M . Suppose that λ is nearly multiplicative. Then, for any*

choice of a normalized left Haar system ν on Γ , the sequence of successive averaging iterates of λ obtained by recursive application of the formula (30)

$$\hat{\lambda}^{(0)} := \lambda, \hat{\lambda}^{(1)} := \hat{\lambda}, \dots, \hat{\lambda}^{(i+1)} := \widehat{(\hat{\lambda}^{(i)})}, \dots \in \text{Psr}_u^p(\Gamma; E)$$

is convergent within the Fréchet space $\text{Psr}^p(\Gamma; E)$ (relative to the C^p -topology; compare Appendix A) to a unique representation $\hat{\lambda}^{(\infty)}$ of class C^p .

By way of preparation, let us introduce some auxiliary terminology. By an *approximation domain* for λ we shall mean a non-empty, invariant, open subset $V = \Gamma V$ of M such that the inequality (40) in Definition 5.1 is satisfied for some choice of metrics on $E|V$. By an *averaging domain* for λ relative to ν we shall mean a non-empty, relatively compact, ν -adapted, open subset U of M whose closure \overline{U} is contained within some approximation domain for λ . (For the notion of adapted set, cf. Appendix B.) The key property of averaging domains—and, more generally, adapted sets—is expressed by the commutativity of the following diagram, where $\nu|U$ denotes the normalized left Haar system on $\Gamma|U \rightrightarrows U$ induced by ν (cf. Proposition B.11), $\text{avg}[\nu]$ and $\text{avg}[\nu|U]$ denote the averaging operators on invertible pseudo-representations associated respectively to ν and $\nu|U$ by means of the formula (30), and res denotes the restriction map.

$$\begin{array}{ccc} \text{Psr}_*^p(\Gamma; E) & \xrightarrow{\text{avg}[\nu]} & \text{Psr}_u^p(\Gamma; E) \\ \downarrow \text{res} & & \downarrow \text{res} \\ \text{Psr}_*^p(\Gamma|U; E|U) & \xrightarrow{\text{avg}[\nu|U]} & \text{Psr}_u^p(\Gamma|U; E|U) \end{array} \quad (41)$$

(The commutativity of this diagram follows from a straightforward computation of the same kind of that carried out in the first paragraph of the proof of Proposition B.13.)

The proof of Theorem 5.2 will occupy the rest of the section. For reasons of length, it will be subdivided into several steps. It will be devoted, essentially in its entirety, to showing that an arbitrary averaging domain U for λ (relative to the chosen normalized left Haar system ν on Γ) has the following two properties:

- (A) For every finite order of differentiability $r \leq p$, the sequence of pseudo-representations $\{\hat{\lambda}^{(i)}|U\}_{i=0}^\infty$ is Cauchy within the Fréchet space $\text{Psr}^r(\Gamma|U; E|U)$.
- (B) There exist, on the restriction of E over U , vector bundle metrics ϕ for which the limit $\lim_i c_{U,\phi}(\hat{\lambda}^{(i)})$ is zero.

As the reader may have noticed, Property (B) is an immediate consequence of the definitions, of the commutativity of (41), and of Lemma 4.8. Property (A) is much more laborious to establish. Once the properties in question are proven, our theorem will be proven as well. Indeed, it is not difficult to see that the sets Γ_U^U corresponding to all possible averaging domains U for λ form an open cover of the manifold Γ . Take all those standard trivializing charts (φ, τ) for the vector bundle $L(s^*E, t^*E)$ (over Γ) whose domains are contained within some of the open sets of this cover. (For the notion of ‘standard trivializing chart’, cf. Appendix A.) The associated seminorms $p_r^{\tau, \varphi, \overline{B}^\varphi}$ [where $\overline{B}^\varphi = \varphi^{-1}(\overline{B}_1(0))$ and $0 \leq r \leq p$] still generate the C^p -topology on the vector space

$\Gamma^p(\Gamma; L(s^*E, t^*E)) = \text{Psr}^p(\Gamma; E)$. Since by (A) the sequence $\{\hat{\lambda}^{(i)}\}$ is Cauchy with respect to every such seminorm, the same sequence will be Cauchy in the Fréchet space $\text{Psr}^p(\Gamma; E)$ and hence will converge to a unique pseudo-representation $\hat{\lambda}^{(\infty)} \in \text{Psr}^p(\Gamma; E)$ of class C^p . Next, since C^p -convergence implies pointwise convergence, for each arrow $g \in \Gamma$ the sequence of linear maps $\{\hat{\lambda}^{(i)}(g)\}$ will be convergent to $\hat{\lambda}^{(\infty)}(g)$ in the finite-dimensional vector space $L(E_{sg}, E_{tg})$ (with respect to whatever norm on that space). It follows that $\hat{\lambda}_{1x}^{(\infty)} = \lim_i \hat{\lambda}_{1x}^{(i)} = id_{E_x}$ for all x in M since every pseudo-representation $\hat{\lambda}^{(i)}$ is unital. Moreover for each composable pair of arrows $g', g \in \Gamma$ we must be able to find some averaging domain U for λ such that $g', g \in \Gamma|U$ and some metric ϕ such that (B) holds, in which case we have the following inequality (the norms appearing below depend on ϕ as described in 4.2, but here for conciseness' sake we add no subscripts)

$$\begin{aligned} \|\hat{\lambda}_{g'g}^{(\infty)} - \hat{\lambda}_{g'}^{(\infty)}\hat{\lambda}_g^{(\infty)}\| &\leq \|\hat{\lambda}_{g'g}^{(\infty)} - \hat{\lambda}_{g'g}^{(i)}\| + \|\hat{\lambda}_{g'g}^{(i)} - \hat{\lambda}_{g'}^{(i)}\hat{\lambda}_g^{(i)}\| \\ &\quad + \|\hat{\lambda}_{g'}^{(i)} - \hat{\lambda}_{g'}^{(\infty)}\| \|\hat{\lambda}_g^{(i)}\| + \|\hat{\lambda}_{g'}^{(\infty)}\| \|\hat{\lambda}_g^{(i)} - \hat{\lambda}_g^{(\infty)}\|, \end{aligned}$$

whose right-hand side becomes arbitrarily small when i is taken large enough. It is thus established that $\hat{\lambda}^{(\infty)}$ is a representation.

Proof, Step I.

So, suppose we are assigned an arbitrary averaging domain U for λ (relative to the given normalized left Haar system ν on Γ). Let us fix some approximation domain V for λ so that $V \supset \overline{U}$. Let us also fix some metric ψ on $E|V$ for which the near multiplicativity inequality (40) is satisfied. Let ϕ denote the restriction of ψ to $E|U$. Put $\Omega = \Gamma_U^U$. These notations will not change throughout the rest of the section. Notice that by the properness of Γ the closure $\overline{\Omega}$ of Ω in Γ must be compact, because

$$\overline{\Omega} \subset s^{-1}(\overline{U}) \cap t^{-1}(\overline{U}) = (s, t)^{-1}(\overline{U} \times \overline{U}).$$

Obviously $\overline{\Omega}$ must be contained within the open set $\Gamma_V^V (= \Gamma^V = \Gamma_V)$. The closure of $\Omega_2 := \Omega \times_t \Omega$ in $\Gamma_2 := \Gamma \times_t \Gamma$ and that of $\Omega_3 := \Omega \times_t \Omega \times_t \Omega$ in $\Gamma_3 := \Gamma \times_t \Gamma \times_t \Gamma$

$$\begin{aligned} \overline{\Omega}_2 &:= \overline{\Omega \times_t \Omega} \subset \overline{\Omega} \times_t \overline{\Omega} \\ \overline{\Omega}_3 &:= \overline{\Omega \times_t \Omega \times_t \Omega} \subset \overline{\Omega} \times_t \overline{\Omega} \times_t \overline{\Omega} \end{aligned}$$

must be compact too. Obviously $\overline{\Omega}_2$ must lie within the open set $\Gamma_V^V \times_t \Gamma_V$.

Let us choose any three standard normed atlases, \mathcal{A}_1 for $L(s^*E, t^*E)$ over $\overline{\Omega}$, $\tilde{\mathcal{A}}_1$ for $L(t^*E, s^*E)$ over $\overline{\Omega}$, and \mathcal{A}_2 for $L(s_2^*E, t_2^*E)$ over $\overline{\Omega}_2$. [Cf. Appendix A. As usual, s_2 and t_2 denote the two maps of Γ_2 into M given respectively by $(g', g) \mapsto sg$ and $\mapsto tg'$.] We can choose \mathcal{A}_1 (resp. $\tilde{\mathcal{A}}_1, \mathcal{A}_2$) so that the domains of the standard normed charts that compose it be contained within Γ_V (resp. $\Gamma_V, \Gamma_V^V \times_t \Gamma_V$). We can further assume that for each standard normed chart in \mathcal{A}_1 the corresponding continuous vector bundle norm is induced by the continuous vector bundle norm on $L(s^*E, t^*E)|\Gamma_V \cong L(s_V^*(E|V), t_V^*(E|V))$ associated in the ordinary way—viz. as described in the paragraph ‘Examples’ following the proof of Lemma A.2 in Appendix A—to the two pullback metrics $s_V^*\psi$ [on $s_V^*(E|V)$] and $t_V^*\psi$ [on $t_V^*(E|V)$] where s_V and t_V denote, respectively,

the restriction of s to Γ_V and the restriction of t to Γ_V . Similarly for $\tilde{\mathcal{A}}_1$ and \mathcal{A}_2 . Having made all these choices, for every natural number r we set [compare (64)]

$$\mathbf{p}_r(\cdot) = \|\cdot\|_{C^r\overline{\mathcal{Q}}; L(s^*E, t^*E), \mathcal{A}_1} \quad (42a)$$

$$\tilde{\mathbf{p}}_r(\cdot) = \|\cdot\|_{C^r\overline{\mathcal{Q}}; L(t^*E, s^*E), \tilde{\mathcal{A}}_1} \quad (42b)$$

$$\mathbf{q}_r(\cdot) = \|\cdot\|_{C^r\overline{\mathcal{Q}}_2; L(s_2^*E, t_2^*E), \mathcal{A}_2} \quad (42c)$$

and, for every $r \leq p$ and for all C^p pseudo-representations $\zeta \in \Gamma^p(\Gamma; L(s^*E, t^*E))$ and $\eta \in \Gamma^p(\Gamma; L(t^*E, s^*E))$,

$$b^{(r)}(\zeta) = \mathbf{p}_r(\text{res}_{\overline{\mathcal{Q}}}^{\Gamma} \zeta) \quad (43a)$$

$$\tilde{b}^{(r)}(\eta) = \tilde{\mathbf{p}}_r(\text{res}_{\overline{\mathcal{Q}}}^{\Gamma} \eta) \quad (43b)$$

$$c^{(r)}(\zeta) = \mathbf{q}_r(\text{res}_{\overline{\mathcal{Q}}_2}^{\Gamma_2} [m^* \zeta - pr_1^* \zeta \circ pr_2^* \zeta]) \quad (43c)$$

where $m, pr_1, pr_2 : \Gamma_2 \rightarrow \Gamma$ stand, respectively, for arrow composition, first, and second projection, and where the expression ‘ $[\cdot \dots \cdot]$ ’ (by abuse of notation) is supposed to indicate the difference between the following two composite vector bundle morphisms.

$$\begin{aligned} s_2^*E &\cong pr_2^*s^*E \cong m^*s^*E \xrightarrow{m^*\zeta} m^*t^*E \cong pr_1^*t^*E \cong t_2^*E \\ s_2^*E &\cong pr_2^*s^*E \xrightarrow{pr_2^*\zeta} pr_2^*t^*E \cong pr_1^*s^*E \xrightarrow{pr_1^*\zeta} pr_1^*t^*E \cong t_2^*E \end{aligned}$$

Notice that our selection criteria for the standard normed atlases \mathcal{A}_1 and \mathcal{A}_2 ensure that the quantities $b^{(0)}(\zeta)$ and $c^{(0)}(\zeta)$ coincide respectively with $b_{U,\phi}(\zeta)$ and $c_{U,\phi}(\zeta)$ as defined at the very beginning of this section.

Proof, Step II.

Throughout the present stage of the proof, r will denote a fixed natural number. Let ζ be a variable ranging over $\Gamma^{r+1}(\Gamma; \text{Lis}(s^*E, t^*E))$ (= invertible C^{r+1} pseudo-representations of Γ on E). By the equation (31b) we have

$$\mathbf{p}_{r+1}(\text{res}_{\overline{\mathcal{Q}}}^{\Gamma} \hat{\zeta}) \leq \mathbf{p}_{r+1}(\text{res}_{\overline{\mathcal{Q}}}^{\Gamma} \zeta) + \mathbf{p}_{r+1}(\text{res}_{\overline{\mathcal{Q}}}^{\Gamma} \langle [a^* \Delta^{\zeta}] \rangle, d_1 \nu)$$

where: (i) Δ^{ζ} is given by (29); (ii) a denotes the diffeomorphism $\Gamma_2 \xrightarrow{\sim} \Gamma_{\div}$ given by $(g, k) \mapsto (gk, k)$; (iii) $[a^* \Delta^{\zeta}]$ indicates the C^{r+1} cross-section of the vector bundle $pr_1^*L(s^*E, t^*E)$ (over the manifold Γ_2) which corresponds to the pullback cross-section $a^* \Delta^{\zeta} \in \Gamma^{r+1}(\Gamma_2; a^*L(s_{\div}^*E, t_{\div}^*E))$ under the vector bundle isomorphism $a^*L(s_{\div}^*E, t_{\div}^*E) \cong L(a^*s_{\div}^*E, a^*t_{\div}^*E) \cong L(pr_1^*s^*E, pr_1^*t^*E) \cong pr_1^*L(s^*E, t^*E)$; (iv) $d_1 \nu$ denotes the “functional” (read: “continuous linear operator”)

$$d_{s; L(s^*E, t^*E)}^{r+1} \nu : \Gamma^{r+1}(\Gamma_2; pr_1^*L(s^*E, t^*E)) \longrightarrow \Gamma^{r+1}(\Gamma; L(s^*E, t^*E))$$

which, as described in the statement of Proposition B.12, arises by integration against the normalized left Haar system ν . In the notations of Proposition B.13, we have

$$\text{res}_{\overline{\mathcal{Q}}}^{\Gamma} \circ d_{s; L(s^*E, t^*E)}^{r+1} \nu = d_{s; \Omega \rightarrow U; L(s^*E, t^*E)}^{r+1} \nu \circ \text{res}_{\overline{\mathcal{Q}}_2}^{\Gamma_2}; \quad (44)$$

this equation follows directly from the commutativity of the diagram (77). In the notations introduced before Lemma A.8, making use of (44) and of the continuity of the “localized” integration functional $d_{s:\Omega \rightarrow U; L(s^*E, t^*E)}^{r+1} \nu$ (Proposition B.13), we get

$$\|res_{\overline{\Omega}}^{\Gamma} \llbracket a^* \Delta^\zeta \rrbracket, d_1 \nu\|_{C^{r+1} \overline{\Omega}; L(s^*E, t^*E)} \leq \|res_{\overline{\Omega}_2}^{\Gamma_2} \llbracket a^* \Delta^\zeta \rrbracket\|_{C^{r+1} \overline{\Omega}_2; pr_1^* L(s^*E, t^*E)}. \quad (45)$$

Since the cross-section $\llbracket a^* \Delta^\zeta \rrbracket$ of the vector bundle $pr_1^* L(s^*E, t^*E)$ corresponds under the canonical vector bundle isomorphism $pr_1^* L(s^*E, t^*E) \cong L(pr_1^* s^*E, pr_1^* t^*E) \cong L(pr_2^* t^*E, t_2^*E)$ to the cross-section of $L(pr_2^* t^*E, t_2^*E)$ arising as the composition $\llbracket m^* \zeta - pr_1^* \zeta \circ pr_2^* \zeta \rrbracket \circ \llbracket pr_2^* \zeta^{-1} \rrbracket$ where $\llbracket pr_2^* \zeta^{-1} \rrbracket$ denotes the cross-section of $L(pr_2^* t^*E, s_2^*E) \cong L(pr_2^* t^*E, pr_2^* s^*E) \cong pr_2^* L(t^*E, s^*E)$ corresponding to $pr_2^* \zeta^{-1}$, making use of the lemmas A.6 and A.7, of the identities (65) and (66), and of the lemmas A.8 and A.9 we derive the following estimate for the right-hand side of (45):

$$\begin{aligned} &= \|res_{\overline{\Omega}_2}^{\Gamma_2} \llbracket m^* \zeta - pr_1^* \zeta \circ pr_2^* \zeta \rrbracket \circ res_{\overline{\Omega}_2}^{\Gamma_2} \llbracket pr_2^* \zeta^{-1} \rrbracket\|_{C^{r+1} \overline{\Omega}_2; L(pr_2^* t^*E, t_2^*E)} \\ \text{[by (67b):]} &\leq \|res_{\overline{\Omega}_2}^{\Gamma_2} \llbracket m^* \zeta - pr_1^* \zeta \circ pr_2^* \zeta \rrbracket\|_{C^r \overline{\Omega}_2; L(s_2^*E, t_2^*E)} \\ &\quad \times \|res_{\overline{\Omega}_2}^{\Gamma_2} \llbracket pr_2^* \zeta^{-1} \rrbracket\|_{C^{r+1} \overline{\Omega}_2; L(pr_2^* t^*E, s_2^*E)} \\ &\quad + \|res_{\overline{\Omega}_2}^{\Gamma_2} \llbracket m^* \zeta - pr_1^* \zeta \circ pr_2^* \zeta \rrbracket\|_{C^{r+1} \overline{\Omega}_2; L(s_2^*E, t_2^*E)} \\ &\quad \times \|res_{\overline{\Omega}_2}^{\Gamma_2} \llbracket pr_2^* \zeta^{-1} \rrbracket\|_{C^r \overline{\Omega}_2; L(pr_2^* t^*E, s_2^*E)} \\ &= \|\dots\|_{C^r} \|res_{\overline{\Omega}_2}^{\Gamma_2} pr_2^* \zeta^{-1}\|_{C^{r+1} \overline{\Omega}_2; pr_2^* L(t^*E, s^*E)} \\ &\quad + \|\dots\|_{C^{r+1}} \|res_{\overline{\Omega}_2}^{\Gamma_2} pr_2^* \zeta^{-1}\|_{C^r \overline{\Omega}_2; pr_2^* L(t^*E, s^*E)} \\ &\leq \|\dots\|_{C^r} \|res_{\overline{\Omega}}^{\Gamma} \zeta^{-1}\|_{C^{r+1} \overline{\Omega}; L(t^*E, s^*E)} + \|\dots\|_{C^{r+1}} \|res_{\overline{\Omega}}^{\Gamma} \zeta^{-1}\|_{C^r \overline{\Omega}; L(t^*E, s^*E)} \\ \text{[by (71):]} &\leq \|\dots\|_{C^r} \|res_{\overline{\Omega}}^{\Gamma} \zeta^{-1}\|_{C^r \overline{\Omega}; L(t^*E, s^*E)}^2 \|res_{\overline{\Omega}}^{\Gamma} \zeta\|_{C^{r+1} \overline{\Omega}; L(s^*E, t^*E)} \\ &\quad + \|\dots\|_{C^{r+1}} \|res_{\overline{\Omega}}^{\Gamma} \zeta^{-1}\|_{C^r}. \end{aligned}$$

Conclusion. *There exists some constant $B_r > 0$ such that for all invertible C^{r+1} pseudo-representations $\zeta \in \Gamma^{r+1}(\Gamma; \text{Lis}(s^*E, t^*E))$ the following inequality holds.*

$$b^{(r+1)}(\hat{\zeta}) \leq b^{(r+1)}(\zeta) + B_r \cdot [(\tilde{b}^{(r)}(\zeta^{-1}))^2 b^{(r+1)}(\zeta) c^{(r)}(\zeta) + \tilde{b}^{(r)}(\zeta^{-1}) c^{(r+1)}(\zeta)] \quad (46)$$

We proceed to obtain a similar estimate for the quantity $c^{(r+1)}(\hat{\zeta})$. Consider the following differentiable mappings.

$$\begin{aligned} pr_{12} : \Gamma_3 &\rightarrow \Gamma_2 & (g', g, k) &\mapsto (g', g) \\ pr_{23} : \Gamma_3 &\rightarrow \Gamma_2 & (g', g, k) &\mapsto (g, k) \\ m_{23} : \Gamma_3 &\rightarrow \Gamma_2 & (g', g, k) &\mapsto (g', gk) \end{aligned}$$

By the formula (31a), since in virtue of the left invariance of the Haar system ν the double integral term in that formula can be rewritten as

$$\begin{aligned} &\int_{tk=sg} \left[\int_{th=sg} \Delta(g'gh, gh) d\nu_{sg}(h) \right] \circ \Delta(gk, k) d\nu_{sg}(k) = \\ &= \int_{tk=sg} \left[\int_{tk'=sg'} \Delta(g'k', k') d\nu_{sg'}(k') \right] \circ \Delta(gk, k) d\nu_{sg}(k) \end{aligned}$$

$$= \left[\int_{tk'=sg'} \Delta(g'k', k') d\nu_{sg'}(k') \right] \circ \left[\int_{tk=sg} \Delta(gk, k) d\nu_{sg}(k) \right],$$

we have the inequality

$$\begin{aligned} \mathbf{q}_{r+1}(\text{res}_{\Omega_2}^{\Gamma_2} [m^* \hat{\zeta} - pr_1^* \hat{\zeta} \circ pr_2^* \hat{\zeta}]) &\leq \\ &\leq \mathbf{q}_{r+1}(\text{res}_{\Omega_2}^{\Gamma_2} \langle [m_{23}^* a^* \Delta^\zeta \circ pr_{23}^* a^* \Delta^\zeta], d_2 \nu \rangle) \\ &\quad + \mathbf{q}_{r+1}(\text{res}_{\Omega_2}^{\Gamma_2} [pr_1^* \langle [a^* \Delta^\zeta], d_1 \nu \rangle] \circ \text{res}_{\Omega_2}^{\Gamma_2} [pr_2^* \langle [a^* \Delta^\zeta], d_1 \nu \rangle]) \end{aligned} \quad (47)$$

where: (i*) by abuse of notation, the term $[m_{23}^* a^* \Delta^\zeta \circ pr_{23}^* a^* \Delta^\zeta]$ denotes the unique cross-section of the vector bundle $pr_{12}^* L(s_2^* E, t_2^* E)$ (over the manifold Γ_3) that corresponds to the composite vector bundle morphism

$$pr_{12}^* t_2^* E \cong m_{23}^* a^* t_2^* E \xleftarrow{m_{23}^* a^* \Delta^\zeta} m_{23}^* a^* s_2^* E \cong pr_{23}^* a^* t_2^* E \xleftarrow{pr_{23}^* a^* \Delta^\zeta} pr_{23}^* a^* s_2^* E \cong pr_{12}^* s_2^* E;$$

(ii*) $d_2 \nu$ stands for the integration functional

$$d_{s_2; L(s_2^* E, t_2^* E)}^{r+1} \nu : \Gamma^{r+1}(\Gamma_3; pr_{12}^* L(s_2^* E, t_2^* E)) \longrightarrow \Gamma^{r+1}(\Gamma_2; L(s_2^* E, t_2^* E));$$

(iii*) the two terms $[pr_1^* \langle \dots \rangle]$ and $[pr_2^* \langle \dots \rangle]$, respectively, denote the following two vector bundle morphisms.

$$\begin{aligned} t_2^* E &\cong pr_1^* t^* E \xleftarrow{pr_1^* \langle \dots \rangle} pr_1^* s^* E \cong (s \circ pr_1)^* E \\ (t \circ pr_2)^* E &\cong pr_2^* t^* E \xleftarrow{pr_2^* \langle \dots \rangle} pr_2^* s^* E \cong s_2^* E \end{aligned}$$

By analogy with the derivation of the above estimate (45), we make use of the identity

$$\text{res}_{\Omega_2}^{\Gamma_2} \circ d_{s_2; L(s_2^* E, t_2^* E)}^{r+1} \nu = d_{s_2; \Omega_2 \rightarrow U; L(s_2^* E, t_2^* E)}^{r+1} \nu \circ \text{res}_{\Omega_3}^{\Gamma_3}$$

and of the continuity of the “localized” integration functional $d_{s_2; \Omega_2 \rightarrow U; L(s_2^* E, t_2^* E)}^{r+1} \nu$ (compare Proposition B.13) in order to deduce the following estimate:

$$\|\text{res}_{\Omega_2}^{\Gamma_2} \langle [m_{23}^* a^* \Delta^\zeta \circ pr_{23}^* a^* \Delta^\zeta], d_2 \nu \rangle\|_{C^{r+1}} \leq \|\text{res}_{\Omega_3}^{\Gamma_3} [m_{23}^* a^* \Delta^\zeta \circ pr_{23}^* a^* \Delta^\zeta]\|_{C^{r+1}}.$$

By Lemma A.6, A.7 and A.8, we see that the right-hand member must be \leq than

$$\begin{aligned} &\|\text{res}_{\Omega_3}^{\Gamma_3} m_{23}^* a^* \Delta^\zeta\|_{C^r} \|\text{res}_{\Omega_3}^{\Gamma_3} pr_{23}^* a^* \Delta^\zeta\|_{C^{r+1}} + \|\text{res}_{\Omega_3}^{\Gamma_3} m_{23}^* a^* \Delta^\zeta\|_{C^{r+1}} \|\text{res}_{\Omega_3}^{\Gamma_3} pr_{23}^* a^* \Delta^\zeta\|_{C^r} \leq \\ &\leq \|\text{res}_{\Omega_2}^{\Gamma_2} a^* \Delta^\zeta\|_{C^r} \|\text{res}_{\Omega_2}^{\Gamma_2} a^* \Delta^\zeta\|_{C^{r+1}} + \|\text{res}_{\Omega_2}^{\Gamma_2} a^* \Delta^\zeta\|_{C^{r+1}} \|\text{res}_{\Omega_2}^{\Gamma_2} a^* \Delta^\zeta\|_{C^r} \\ &= \|\text{res}_{\Omega_2}^{\Gamma_2} [a^* \Delta^\zeta]\|_{C^r} \|\text{res}_{\Omega_2}^{\Gamma_2} [a^* \Delta^\zeta]\|_{C^{r+1}}. \end{aligned}$$

The second factor can be estimated as after (45), whilst for the first one we have

$$\|\text{res}_{\Omega_2}^{\Gamma_2} [a^* \Delta^\zeta]\|_{C^r} \leq \|\text{res}_{\Omega_2}^{\Gamma_2} [m^* \zeta - pr_1^* \zeta \circ pr_2^* \zeta]\|_{C^r} \|\text{res}_{\Omega}^{\Gamma} \zeta^{-1}\|_{C^r}. \quad (48)$$

[For $r = 0$ this is an instance of the estimate (67a), whereas for $r = l + 1$ it is implied by the sharper estimate (67b) since $\|\cdot\|_{C^l} \leq \|\cdot\|_{C^{l+1}}$.] Estimation of the second summand in the right-hand side of the inequality (47) goes even faster:

$$\|\text{res}_{\Omega_2}^{\Gamma_2} [pr_1^* \langle \dots \rangle] \circ \text{res}_{\Omega_2}^{\Gamma_2} [pr_2^* \langle \dots \rangle]\|_{C^{r+1}} \leq$$

$$\begin{aligned}
&\leq \|res_{\Omega_2}^{F_2} [pr_1^* \langle \dots \rangle]\|_{C^r} \|res_{\Omega_2}^{F_2} [pr_2^* \langle \dots \rangle]\|_{C^{r+1}} + \|res_{\Omega_2}^{F_2} [pr_1^* \langle \dots \rangle]\|_{C^{r+1}} \|res_{\Omega_2}^{F_2} [pr_2^* \langle \dots \rangle]\|_{C^r} \\
&\leq \|res_{\Omega}^{F_2} \langle \dots \rangle\|_{C^r} \|res_{\Omega}^{F_2} \langle \dots \rangle\|_{C^{r+1}} + \|res_{\Omega}^{F_2} \langle \dots \rangle\|_{C^{r+1}} \|res_{\Omega}^{F_2} \langle \dots \rangle\|_{C^r} \\
&= \|res_{\Omega}^{F_2} \langle \dots \rangle\|_{C^r} \|res_{\Omega}^{F_2} \langle \dots \rangle\|_{C^{r+1}}.
\end{aligned}$$

The second factor can be dealt with as before, whereas the first factor must be \leq than the right-hand member of (48).

Conclusion. *There must be some constant $C_r > 0$ such that the following inequality holds for all invertible C^{r+1} pseudo-representations $\zeta \in \Gamma^{r+1}(\Gamma; \text{Lis}(s^*E, t^*E))$.*

$$c^{(r+1)}(\hat{\zeta}) \leq C_r \cdot [(\tilde{b}^{(r)}(\zeta^{-1}))^3 b^{(r+1)}(\zeta) c^{(r)}(\zeta) + (\tilde{b}^{(r)}(\zeta^{-1}))^2 c^{(r+1)}(\zeta)] c^{(r)}(\zeta) \quad (49)$$

Proof, Step III.

Next, let us go back to the given nearly multiplicative C^p pseudo-representation $\lambda \in \Gamma^p(\Gamma; L(s^*E, t^*E))$. For each iteration index i , the i th averaging iterate $\hat{\lambda}^{(i)}$ is itself a nearly multiplicative (hence invertible) pseudo-representation of class C^p . For every finite order of differentiability $r \leq p$, let us introduce the following quantities.

$$b_i^{(r)} = b^{(r)}(\hat{\lambda}^{(i)}) \quad (50a)$$

$$\tilde{b}_i^{(r)} = \tilde{b}^{(r)}((\hat{\lambda}^{(i)})^{-1}) \quad (50b)$$

$$c_i^{(r)} = c^{(r)}(\hat{\lambda}^{(i)}) \quad (50c)$$

Also, let us put $\varepsilon = 6b_{U,\phi}(\lambda)^2 c_{U,\phi}(\lambda) (\leq \frac{2}{3} < 1)$. We want to show that the four assertions below are true for every natural number $r \leq p$. In order to accomplish this goal, we shall be arguing by induction on r .

S1(r). *The sequence $\{b_i^{(r)}\}_{i=0}^\infty$ is bounded.*

S2(r). *The sequence $\{\tilde{b}_i^{(r)}\}_{i=0}^\infty$ is bounded.*

S3(r). *There exists some constant $R_r > 0$ such that $c_{i+1}^{(r)} \leq R_r \cdot (c_i^{(r)})^2$ for all $i \in \mathbb{N}$.*

S4(r). *There exists some index $i_r \in \mathbb{N}$ such that $c_i^{(r)} \leq \varepsilon^{2^{(i-i_r)}}$ for all $i \geq i_r$.*

For $r = 0$, all of these assertions are true. Indeed, to begin with, we know that for every unital pseudo-representation $\zeta \in \text{Psu}_u^0(\Gamma; E)$ which satisfies the near multiplicativity inequality $c_{U,\phi}(\zeta) \leq \frac{1}{9} b_{U,\phi}(\zeta)^{-2}$ one must have $b^{(0)}(\zeta) = b_{U,\phi}(\zeta) \geq 1$ and therefore $c^{(0)}(\zeta) = c_{U,\phi}(\zeta) < 1$. Thus, by the estimate (34a),

$$\tilde{b}^{(0)}(\zeta^{-1}) \leq b^{(0)}(\zeta)/(1 - c^{(0)}(\zeta)). \quad (51)$$

Secondly, we know that the iterates $\hat{\lambda}^{(i)}$ are unital and satisfy the near multiplicativity inequality. Thirdly, by virtue of the commutativity of the diagram (41), and by the estimates (35) referring to the groupoid $\Gamma \mid U \rightrightarrows U$ and the Haar system $\nu \mid U$, the two sequences of non-negative real numbers $\{b_i^{(0)}\}$, $\{c_i^{(0)}\}$ must satisfy the hypotheses of Lemma 4.8, whence the following statements must be true.

$$\mathbf{S1}(0). \quad b_i^{(0)} \leq b_i^{(0)} / (1 - c_i^{(0)}) \leq \sqrt{3} b_0^{(0)} \text{ [by (37b) since } 0 < 1 - c_i^{(0)} \leq 1].$$

$$\mathbf{S2}(0). \quad \tilde{b}_i^{(0)} \leq b_i^{(0)} / (1 - c_i^{(0)}) \leq \sqrt{3} b_0^{(0)} \text{ [by (37b) and (51)].}$$

$$\mathbf{S3}(0). \quad c_{i+1}^{(0)} \leq 2(c_i^{(0)})^2 [b_i^{(0)} / (1 - c_i^{(0)})]^2 \leq 6(b_0^{(0)})^2 (c_i^{(0)})^2 \text{ [by (35b) and (37b)].}$$

$$\mathbf{S4}(0). \quad c_i^{(0)} \leq \varepsilon^{2^i} / [6(b_0^{(0)})^2] < \varepsilon^{2^i} \text{ [by (37a) since } b_0^{(0)} \geq 1].$$

Next, suppose that the statements $\mathbf{S1}(r)$ – $\mathbf{S4}(r)$ are valid for a certain order of differentiability $r \leq p - 1$. By the equations (46) and (49), we know that there must be two positive constants B_r and C_r such that the two inequalities below hold for all $i \in \mathbb{N}$.

$$\begin{aligned} b_{i+1}^{(r+1)} &\leq b_i^{(r+1)} + B_r \cdot [(\tilde{b}_i^{(r)})^2 b_i^{(r+1)} c_i^{(r)} + \tilde{b}_i^{(r)} c_i^{(r+1)}] \\ c_{i+1}^{(r+1)} &\leq C_r \cdot [(\tilde{b}_i^{(r)})^3 b_i^{(r+1)} c_i^{(r)} + (\tilde{b}_i^{(r)})^2 c_i^{(r+1)}] c_i^{(r)} \end{aligned}$$

Now, the inductive assumption $\mathbf{S2}(r)$ entails the existence of a positive constant L_r such that the following two inequalities are satisfied for all $i \in \mathbb{N}$.

$$b_{i+1}^{(r+1)} \leq b_i^{(r+1)} + L_r \cdot [b_i^{(r+1)} c_i^{(r)} + c_i^{(r+1)}] \quad (52a)$$

$$c_{i+1}^{(r+1)} \leq L_r \cdot [b_i^{(r+1)} c_i^{(r)} + c_i^{(r+1)}] c_i^{(r)} \quad (52b)$$

In order to complete the inductive step, we have to “solve” this recursive system.

Proof, Step IV.

Lemma 5.3. *Let $\{c_0, c_1, c_2, \dots\}$, $\{b'_0, b'_1, b'_2, \dots\}$ and $\{c'_0, c'_1, c'_2, \dots\}$ be sequences of non-negative real numbers. Let L, R and $\epsilon < 1$ be positive real numbers. Suppose that*

$$b'_{i+1} \leq b'_i + L \cdot (b'_i c_i + c'_i) \quad (53a)$$

$$c'_{i+1} \leq L \cdot (b'_i c_i + c'_i) c_i \quad (53b)$$

$$c_{i+1} \leq R \cdot c_i^2 \quad (53c)$$

and that $c_i \leq c'_i$ for all i . Furthermore, suppose there is some index I such that $c_i \leq \epsilon^{2^{(i-I)}}$ for all $i \geq I$. Then, the following three statements must be true: (a) the sequence $\{b'_i\}$ is bounded; (b) there exists some constant $R' > 0$ such that $c'_{i+1} \leq R' \cdot (c'_i)^2$ for all i ; (c) there exists some index $I' \geq I$ such that $c'_{i'} \leq \epsilon^{2^{(i'-I')}} for all $i' \geq I'$.$

Proof. At the expense of reindexing the given sequences, it will be no loss of generality to assume that $I = 0$. Under this assumption, for every i we will have $c_i \leq \epsilon^{2^i} \leq \epsilon < 1$ and, therefore, $c_i^2 \leq c_i$. Let us put $a'_i = b'_i c_i + c'_i$. Then

$$\begin{aligned} a'_{i+1} &= b'_{i+1} c_{i+1} + c'_{i+1} \\ &\leq R b'_i c_i^2 + R L a'_i c_i^2 + c'_{i+1} && \text{[by (53a) and (53c)]} \\ &\leq R b'_i c_i^2 + R L a'_i c_i + c'_{i+1} && \text{[because } c_i^2 \leq c_i] \\ &\leq R b'_i c_i^2 + (R + 1) L a'_i c_i && \text{[by (53b)]} \\ &\leq (R + 1) L a'_i c_i + R b'_i c_i^2 + R c'_i c_i && \text{[a fortiori]} \end{aligned}$$

$$\begin{aligned} &= (R+1)La'_i c_i + R \cdot (b'_i c_i + c'_i) c_i \\ &= (RL + L + R)a'_i c_i \end{aligned}$$

and thus, setting $K = RL + L + R$,

$$a'_{i+1} \leq Ka'_i c_i,$$

whence $a'_1 \leq Ka'_0 c_0$, $a'_2 \leq K(Ka'_0 c_0)c_1$, $a'_3 \leq K(K^2 a'_0 c_0 c_1)c_2$, and in general

$$a'_i \leq (b'_0 c_0 + c'_0) K^i \prod_{n=0}^{i-1} c_n. \quad (54)$$

Since $1 + 2 + \dots + 2^{i-1} = 2^i - 1$, it follows from (54) in combination with (53a) that

$$\begin{aligned} b'_{i+1} &\leq b'_i + La'_i \leq b'_i + L \cdot (b'_0 c_0 + c'_0) K^i \epsilon^{1+2+\dots+2^{i-1}} \\ &= b'_i + L\epsilon^{-1} \cdot (b'_0 c_0 + c'_0) K^i \epsilon^{2^i} \end{aligned}$$

and therefore, by induction,

$$b'_i \leq b'_0 + L\epsilon^{-1} \cdot (c_0 b'_0 + c'_0) \sum_{n=0}^{i-1} K^n \epsilon^{2^n}.$$

The last inequality shows that the sequence $\{b'_i\}$ is bounded, which was our first claim (a). Using this fact, in combination with the hypothesis $c_i \leq c'_i$ and with (53b), we are able to establish our second claim (b) as well:

$$c'_{i+1} \leq L \cdot (b'_i c_i + c'_i) c_i \leq L \cdot (b'_i c'_i + c'_i) c'_i = L \cdot (b'_i + 1)(c'_i)^2.$$

As to our third claim (c), notice that

$$\begin{aligned} c'_{i+1} &\leq La'_i c_i && \text{[by (53b)]} \\ &\leq L \cdot (b'_0 c_0 + c'_0) K^i \prod_{n=0}^{i-1} c_n \cdot c_i && \text{[by (54)]} \\ &= L \cdot (b'_0 c_0 + c'_0) K^i \prod_{n=0}^i c_n \\ &\leq L \cdot (b'_0 c_0 + c'_0) K^i \epsilon^{1+2+\dots+2^i} && \text{[because } c_n \leq \epsilon^{2^n}] \\ &= [L\epsilon^{-1} \cdot (b'_0 c_0 + c'_0) K^i \epsilon^{2^i}] \epsilon^{2^i}. \end{aligned}$$

Since $\lim_i K^i \epsilon^{2^i} = 0$, the factor within brackets will be < 1 if we take i large enough. \square

Proof, Step V.

By applying the preceding lemma to the recursive system of inequalities (52) and to the inductive hypotheses S3(r) and S4(r), we immediately deduce the truth of the assertions S1($r+1$), S3($r+1$), and S4($r+1$). From (71), we deduce an inequality of the form

$$\tilde{b}^{(r+1)}((\hat{\lambda}^{(i)})^{-1}) \leq \text{const} \cdot \tilde{b}^{(r)}((\hat{\lambda}^{(i)})^{-1})^2 b^{(r+1)}(\hat{\lambda}^{(i)}),$$

so the remaining statement S2($r+1$) follows from the inductive hypothesis S2(r) and from the (already proven) statement S1($r+1$). This concludes the inductive step.

Proof, Step VI.

On the basis of the relation (31b) and of the estimates derived in Step II, for each order of differentiability $r \geq 1$ such that λ is of class C^r (the case $r = 0$ being left to the reader), and for every iteration index i , we have the following estimate:

$$\begin{aligned} p_r(\text{res}_{\overline{\Omega}}^{\Gamma} \hat{\lambda}^{(i+1)} - \text{res}_{\overline{\Omega}}^{\Gamma} \hat{\lambda}^{(i)}) &\leq B_{r-1} \cdot [(\tilde{b}_i^{(r-1)})^2 b_i^{(r)} c_i^{(r-1)} + \tilde{b}_i^{(r-1)} c_i^{(r)}] \\ &\leq B_{r-1} \sup_n \{(\tilde{b}_n^{(r-1)})^2 b_n^{(r)} + \tilde{b}_n^{(r-1)}\} \cdot c_i^{(r)} \end{aligned}$$

[in deriving it, we use the fact that $c_i^{(r-1)} \leq c_i^{(r)}$, which is an immediate consequence of the obvious inequality $\mathbf{q}_{r-1}(-) \leq \mathbf{q}_r(-)$.] Since by S4(r) we have $c_i^{(r)} \leq \varepsilon^{2(i-r)}$ for $i \geq i_r$, we conclude that the sequence $\{\text{res}_{\overline{\Omega}}^{\Gamma} \hat{\lambda}^{(i)}\}$ is Cauchy within $\Gamma^r(\overline{\Omega}; L(s^*E, t^*E))$ relative to the C^r -norm topology. Corollary A.5 then implies that the sequence $\{\text{res}_{\overline{\Omega}}^{\Gamma} \hat{\lambda}^{(i)} = \text{res}_{\Omega}^{\Gamma} \hat{\lambda}^{(i)}|_U\}$ is Cauchy within $\Gamma^r(\Omega; L(s^*E, t^*E))$ relative to the C^r -topology, which is the desired property (A) for the given averaging domain U . The property (B) is an immediate consequence of S4(0). Our theorem is thus proven.

6. Fast convergence theorem II (connections)

Theorem 6.1. *Let Γ be a proper Lie groupoid. Let $\Psi \in \text{Conn}_u^p(\Gamma)$ be a unital groupoid connection of class C^p ($p = 0, 1, 2, \dots, \infty$) on Γ . Suppose that Ψ is nearly multiplicative (in the sense of Definition 5.1). Then the sequence of all successive averaging iterates of Ψ computed with respect to any given normalized left Haar system ν on Γ by means of the formula (16a)*

$$\hat{\Psi}^{(0)} := \Psi, \quad \hat{\Psi}^{(1)} := \hat{\Psi}, \quad \dots, \quad \hat{\Psi}^{(i+1)} := (\widehat{\hat{\Psi}^{(i)}}), \quad \dots \in \text{Conn}_u^p(\Gamma)$$

converges within the affine Fréchet manifold $\text{Conn}^p(\Gamma)$ (which, by definition, consists of all groupoid connections of class C^p on Γ) to a multiplicative C^p connection $\hat{\Psi}^{(\infty)}$.

Proof. (The reader is referred to Section 3 for all those notations that we will be using without commentary. Familiarity with the proof of Theorem 5.2 is assumed. As in the preceding section, we let M denote the base manifold of Γ .) Let H be an arbitrary non-degenerate connection on $\Gamma \rightrightarrows M$. For each divisible pair of arrows $(g, h) \in \Gamma_{\div}$ and for every tangent vector $v \in T_{sg=sh}M$, the tangent vector $(\eta_g^H v \div \eta_h^H v) - \eta_{gh^{-1}}^H \lambda_h^H v \in T_{gh^{-1}}\Gamma$ must be s -vertical (i.e. must lie within the s -vertical subspace $T_{gh^{-1}}^{\uparrow}\Gamma \subset T_{gh^{-1}}\Gamma$) because

$$\begin{aligned} (T_{gh^{-1}}s)((\eta_g^H v \div \eta_h^H v) - \eta_{gh^{-1}}^H \lambda_h^H v) &= (T_h t) \eta_h^H v - (id_{T_{th}M}) \lambda_h^H v \\ &= \lambda_h^H v - \lambda_h^H v = 0. \end{aligned}$$

As a consequence, we have a linear map

$$(\eta_g^H \div \eta_h^H) - \eta_{gh^{-1}}^H \circ \lambda_h^H \in L(T_{sh}M, T_{gh^{-1}}^{\uparrow}\Gamma). \quad (55)$$

Since H is non-degenerate and since, as we just saw, $T_{gh^{-1}}s \circ (\eta_g^H \div \eta_h^H) = T_h t \circ \eta_h^H = \lambda_h^H$,

$$\Delta^H(g, h) = \omega_{gh^{-1}} \circ \beta_{gh^{-1}}^H \circ \delta^H(g, h)$$

$$\begin{aligned}
&= \omega_{gh^{-1}} \circ (\delta^H(g, h) - \eta_{gh^{-1}}^H \circ T_{gh^{-1}} s \circ \delta^H(g, h)) \\
&= \omega_{gh^{-1}} \circ ([\eta_g^H \div \eta_h^H] \circ (\lambda_h^H)^{-1} - \eta_{gh^{-1}}^H \circ \lambda_h^H \circ (\lambda_h^H)^{-1}) \\
&= \omega_{gh^{-1}} \circ ([\eta_g^H \div \eta_h^H] - \eta_{gh^{-1}}^H \circ \lambda_h^H \circ (\lambda_h^H)^{-1}). \tag{56}
\end{aligned}$$

Therefore, for any other non-degenerate groupoid connection Φ , and for every tangent vector v as above,

$$\begin{aligned}
\Delta^H(g, h)\lambda_h^H v &= \omega_{gh^{-1}}([\eta_g^H v \div \eta_h^H v] - \eta_{gh^{-1}}^H \lambda_h^H v) \\
&= \omega_{gh^{-1}}(\beta_{gh^{-1}}^\Phi([\eta_g^H v \div \eta_h^H v] - \eta_{gh^{-1}}^H \lambda_h^H v)) && \text{[by } s\text{-verticality (55)]} \\
&= pr_1(\pi_{gh^{-1}}^\Phi([\eta_g^H v \div \eta_h^H v] - \eta_{gh^{-1}}^H \lambda_h^H v)) && \text{[by definition (18)]} \\
&= \dot{q}_{g,h}^\Phi(X_g^H v, X_h^H v, v) - X_{gh^{-1}}^H \lambda_h^H v && \text{[see 3.5]} \\
&= \dot{q}_\uparrow^\Phi(g, h)(X_g^H v, X_h^H v) - X_{gh^{-1}}^H \lambda_h^H v + \Delta^\Phi(g, h)\lambda_h^\Phi v. && \text{[by (21) and (25)]} \tag{57}
\end{aligned}$$

Now, under the proviso that $\hat{\Phi}$ be itself non-degenerate, making $H = \hat{\Phi}$ in (57) and referring back to the notational conventions and to the computations in the proof of Proposition 3.7,

$$\begin{aligned}
\Delta^{\hat{\Phi}}(g, h) \circ \hat{\lambda}_h &= \dot{q}_\uparrow(g, h) \circ (\hat{X}_g, \hat{X}_h) - \hat{X}_{gh^{-1}} \circ \hat{\lambda}_h + \Delta^\Phi(g, h) \circ \lambda_h \\
&= \iint \Delta^\Phi(gk, hk) \circ \dot{s}_\uparrow(h) \circ (\Delta^\Phi(hk, k) - \Delta^\Phi(hk', k')) d\nu(k) d\nu(k') \\
&= \int \Delta^\Phi(gk, hk) \circ \Delta^\lambda(hk, k) d\nu(k) - \iint \Delta^\Phi(gk, hk) \circ \Delta^\lambda(hk', k') d\nu(k) d\nu(k') \\
&= \int \Delta^\Phi(gk, hk) \circ \Delta^\lambda(hk, k) d\nu(k) - \int \Delta^\Phi(gk, hk) d\nu(k) \circ \int \Delta^\lambda(hk, k) d\nu(k). \tag{58a}
\end{aligned}$$

This is the analog for groupoid connections of Equation (31a), whereas

$$\begin{aligned}
\hat{\eta}_g^\Phi - \eta_g^\Phi &= \int \delta^\Phi(gk, k) - \eta_g^\Phi d\nu(k) \\
&= \int [\eta_{gk}^\Phi \div \eta_k^\Phi] \circ \lambda_k^{-1} - \eta_g^\Phi d\nu(k) \\
&= \omega_g^{-1} \circ \int \Delta^\Phi(gk, k) d\nu(k) \tag{58b}
\end{aligned}$$

is the analog of Equation (31b).

By an ‘averaging domain for Ψ ’ we shall mean an averaging domain for the effect λ^Ψ of Ψ (relative to ν). As in Step I of the proof of Theorem 5.2, let us consider an arbitrary averaging domain U for Ψ . The subset $\Omega = \Gamma_U^U$ of the arrow manifold Γ will be open and relatively compact. The closure $\overline{\Omega}_\div$ of the open set $\Omega_\div := \Omega \times_s \Omega$ in Γ_\div will also be compact, since it lies within $\overline{\Omega}_s \times_s \overline{\Omega}$. Let us fix an arbitrary natural number, say r . Let Φ denote a variable ranging over the subset of $\text{Conn}_*^r(\Gamma)$ (= non-degenerate C^r groupoid connections on Γ) consisting of all Φ such that $\hat{\Phi}$ also belongs to $\text{Conn}_*^r(\Gamma)$. Recall that by definition Δ^Φ coincides with the composite vector bundle morphism

$$s_\div^* TM \xrightarrow{\delta^\Phi} q_\div^* T\Gamma \xrightarrow{q_\div^* \beta^\Phi} q_\div^* T^\uparrow \Gamma \xrightarrow{q_\div^* \omega} q_\div^* t^* g \cong t_\div^* g$$

(each component of which is at least C^r). Letting $[r_\div^* \hat{\lambda}]$ denote the vector bundle isomorphism

$$r_\div^* s^* TM \xrightarrow{r_\div^* \hat{\lambda}} r_\div^* t^* TM \cong s_\div^* TM, \tag{59}$$

where as before $\hat{\lambda} = (\lambda^\Phi)^\wedge = \lambda^\Phi$, we have the C^r -norm estimate

$$\begin{aligned} \|res_{\overline{\Omega}_\div}^{\Gamma_\div} \Delta^\Phi\|_{C^r} &= \|res_{\overline{\Omega}_\div}^{\Gamma_\div} (\Delta^\Phi \circ [r_\div^* \hat{\lambda}]) \circ res_{\overline{\Omega}_\div}^{\Gamma_\div} [r_\div^* \hat{\lambda}]^{-1}\|_{C^r} \\ &\leq \|res_{\overline{\Omega}_\div}^{\Gamma_\div} (\Delta^\Phi \circ [r_\div^* \hat{\lambda}])\|_{C^r} \|res_{\overline{\Omega}_\div}^{\Gamma_\div} r_\div^* \hat{\lambda}^{-1}\|_{C^r} \\ &\leq \|res_{\overline{\Omega}_\div}^{\Gamma_\div} (\Delta^\Phi \circ [r_\div^* \hat{\lambda}])\|_{C^r} \|res_{\overline{\Omega}_\div}^{\Gamma_\div} \hat{\lambda}^{-1}\|_{C^r}. \end{aligned} \quad (60)$$

The first factor in (60) can be estimated through Equation (58a). In detail, let us rewrite (58a) in “implicit form”, namely,

$$\Delta^\Phi \circ [r_\div^* \hat{\lambda}] = \langle [m_\div^* \Delta^\Phi \circ pr_{2,3}^* a^* \Delta^\lambda], d'_\div \nu \rangle - \langle [m_\div^* \Delta^\Phi], d''_\div \nu \rangle \circ [r_\div^* \langle [a^* \Delta^\lambda], d_1 \nu \rangle]$$

where: (i) m_\div is the mapping of $\Gamma_\div \times_{s \circ r_\div} \Gamma$ into Γ_\div given by $(g, h; k) \mapsto (gk, hk)$; (ii) $pr_{2,3}$ is the mapping of $\Gamma_\div \times_{s \circ r_\div} \Gamma$ into $\Gamma_2 (= \Gamma_s \times_t \Gamma)$ given by $(g, h; k) \mapsto (h, k)$; (iii) $\lambda = \lambda^\Phi$; (iv) $[m_\div^* \Delta^\Phi \circ pr_{2,3}^* a^* \Delta^\lambda]$ denotes the unique cross-section of the vector bundle $pr_\div^* L(r_\div^* s^* TM, t_\div^* \mathfrak{g})$ that corresponds to the composite vector bundle morphism

$$pr_\div^* t_\div^* \mathfrak{g} \cong m_\div^* t_\div^* \mathfrak{g} \xleftarrow{m_\div^* \Delta^\Phi} m_\div^* s_\div^* TM \cong pr_{2,3}^* a^* t_\div^* TM \xleftarrow{pr_{2,3}^* a^* \Delta^\lambda} pr_{2,3}^* a^* s_\div^* TM \cong pr_\div^* r_\div^* s^* TM,$$

$pr_\div : \Gamma_\div \times_{s \circ r_\div} \Gamma \rightarrow \Gamma_\div$ being the projection $(g, h; k) \mapsto (g, h)$; (v) $d'_\div \nu$ is the integration functional

$$d_{s \circ r_\div; L(r_\div^* s^* TM, t_\div^* \mathfrak{g})}^r \nu : \Gamma^r(\Gamma_\div \times_{s \circ r_\div} \Gamma; pr_\div^* L(r_\div^* s^* TM, t_\div^* \mathfrak{g})) \longrightarrow \Gamma^r(\Gamma_\div; L(r_\div^* s^* TM, t_\div^* \mathfrak{g}));$$

(vi) $[m_\div^* \Delta^\Phi]$ denotes the unique cross-section of the vector bundle $pr_\div^* L(s_\div^* TM, t_\div^* \mathfrak{g})$ that corresponds to the vector bundle morphism

$$pr_\div^* t_\div^* \mathfrak{g} \cong m_\div^* t_\div^* \mathfrak{g} \xleftarrow{m_\div^* \Delta^\Phi} m_\div^* s_\div^* TM \cong pr_\div^* s_\div^* TM;$$

(vii) $d''_\div \nu$ is the integration functional

$$d_{s \circ r_\div; L(s_\div^* TM, t_\div^* \mathfrak{g})}^r \nu : \Gamma^r(\Gamma_\div \times_{s \circ r_\div} \Gamma; pr_\div^* L(s_\div^* TM, t_\div^* \mathfrak{g})) \longrightarrow \Gamma^r(\Gamma_\div; L(s_\div^* TM, t_\div^* \mathfrak{g}));$$

(viii) $[r_\div^* \langle \dots \rangle]$ denotes the composite vector bundle morphism obtained by replacing ‘ $\hat{\lambda}$ ’ with ‘ $\langle \dots \rangle$ ’ in (59), where $\langle \dots \rangle = \langle [a^* \Delta^\lambda], d_1 \nu \rangle$ has the same meaning as in Step II of the proof of Theorem 5.2. By making repeated use of estimates of type $\|\vartheta' \circ \vartheta\|_{C^r} \leq \|\vartheta'\|_{C^r} \|\vartheta\|_{C^r}$, we see by computations entirely analogous to those in the proof of Theorem 5.2 that

$$\begin{aligned} \|res_{\overline{\Omega}_\div}^{\Gamma_\div} (\Delta^\Phi \circ [r_\div^* \hat{\lambda}])\|_{C^r} &\leq \|res_{\overline{\Omega}_\div \times_t \Gamma}^{\Gamma_\div \times_t \Gamma} m_\div^* \Delta^\Phi\|_{C^r} \|res_{\overline{\Omega}_\div \times_t \Gamma}^{\Gamma_\div \times_t \Gamma} pr_{2,3}^* a^* \Delta^\lambda\|_{C^r} \\ &\quad + \|res_{\overline{\Omega}_\div \times_t \Gamma}^{\Gamma_\div \times_t \Gamma} m_\div^* \Delta^\Phi\|_{C^r} \|res_{\overline{\Omega}_\div}^{\Gamma_\div} r_\div^* \langle [a^* \Delta^\lambda], d_1 \nu \rangle\|_{C^r} \\ &\leq \|res_{\overline{\Omega}_\div}^{\Gamma_\div} \Delta^\Phi\|_{C^r} (\|res_{\overline{\Omega}_2}^{\Gamma_2} a^* \Delta^\lambda\|_{C^r} + \|res_{\overline{\Omega}_2}^{\Gamma_2} \langle [a^* \Delta^\lambda], d_1 \nu \rangle\|_{C^r}) \\ &\leq \|res_{\overline{\Omega}_\div}^{\Gamma_\div} \Delta^\Phi\|_{C^r} \|res_{\overline{\Omega}_2}^{\Gamma_2} [a^* \Delta^\lambda]\|_{C^r} \\ &\leq \|res_{\overline{\Omega}_\div}^{\Gamma_\div} \Delta^\Phi\|_{C^r} \|res_{\overline{\Omega}_2}^{\Gamma_2} [m^* \lambda - pr_1^* \lambda \circ pr_2^* \lambda]\|_{C^r} \|res_{\overline{\Omega}_2}^{\Gamma_2} \lambda^{-1}\|_{C^r}. \end{aligned} \quad (61)$$

Putting (60) and (61) together, we obtain

$$\begin{aligned} \|res_{\overline{\mathcal{Q}}^\div}^{\Gamma^\div} \Delta^{\hat{\Phi}}\|_{C^r} &\leq \|res_{\overline{\mathcal{Q}}^\div}^{\Gamma^\div} \Delta^{\Phi}\|_{C^r} \\ &\times \|res_{\overline{\mathcal{Q}}_2}^{\Gamma_2} [m^* \lambda^\Phi - pr_1^* \lambda^\Phi \circ pr_2^* \lambda^\Phi]\|_{C^r} \|res_{\overline{\mathcal{Q}}}^{\Gamma} (\lambda^\Phi)^{-1}\|_{C^r} \|res_{\overline{\mathcal{Q}}}^{\Gamma} (\lambda^{\hat{\Phi}})^{-1}\|_{C^r}. \end{aligned} \quad (62)$$

Next, from the identity (58b) we deduce the following estimate for the $\|\cdot\|_{C^r \overline{\mathcal{Q}}}$ -norm of the difference $\hat{\eta}^\Phi - \eta^\Phi \in \Gamma^r(\Gamma; L(s^* TM, T^\dagger \Gamma))$:

$$\begin{aligned} \|res_{\overline{\mathcal{Q}}}^{\Gamma} (\hat{\eta}^\Phi - \eta^\Phi)\|_{C^r} &= \|\omega^{-1} \circ res_{\overline{\mathcal{Q}}}^{\Gamma} \langle [a^* \Delta^\Phi], d_1^c \nu \rangle\|_{C^r} \\ &\leq \|res_{\overline{\mathcal{Q}}^\div}^{\Gamma^\div} \Delta^\Phi\|_{C^r}, \end{aligned} \quad (63)$$

where $d_1^c \nu$ ('c' like 'connection') is the integration functional

$$d_{s; L(s^* TM, t^* g)}^r \nu : \Gamma^r(\Gamma_2; pr_1^* L(s^* TM, t^* g)) \longrightarrow \Gamma^r(\Gamma; L(s^* TM, t^* g)),$$

and the interpretation of the term $[a^* \Delta^\Phi]$ is obvious (by analogy with the term $[a^* \Delta^\lambda]$).

From now on until the end of the section, we shall let λ stand for the near representation λ^Ψ associated to Ψ (that is the effect of Ψ). We shall also let $\hat{\eta}^{(i)}$ ($i = 0, 1, 2, \dots$) denote the horizontal lift corresponding to $\hat{\Psi}^{(i)}$ ($= i$ th averaging iterate of Ψ). Inductive application of the identity (17) yields that for every index i the i th averaging iterate of λ (which, recall, was denoted by $\hat{\lambda}^{(i)}$ in Section 5) coincides with the near representation associated to the i th averaging iterate of Ψ (viz. the effect of $\hat{\Psi}^{(i)}$). Let U, ϕ be as in Step I of the proof of Theorem 5.2 (relative to the near representation $\lambda = \lambda^\Psi$). We carry over all the notations introduced in the course of the proof of that theorem—for example (42), (43) and (50). In addition, we fix an arbitrary standard normed atlas for $L(s^* TM, t^* g)$ over $\overline{\mathcal{Q}}^\div$, say \mathcal{A}_\div , and then, for each natural number $r \leq p$ and for every non-degenerate C^r groupoid connection $\Phi \in \text{Conn}_*^r(\Gamma)$, set

$$d^{(r)}(\Phi) = \|res_{\overline{\mathcal{Q}}^\div}^{\Gamma^\div} \Delta^\Phi\|_{C^r \overline{\mathcal{Q}}^\div; L(s^* TM, t^* g), \mathcal{A}_\div}$$

and, for every index i ,

$$d_i^{(r)} = d^{(r)}(\hat{\Psi}^{(i)}).$$

From the estimate (62) we immediately deduce that there must be some positive constant D_r such that for every index i (notations as in Step III of the proof of Theorem 5.2)

$$d_{i+1}^{(r)} \leq D_r d_i^{(r)} c_i^{(r)} \tilde{b}_i^{(r)} \tilde{b}_{i+1}^{(r)}.$$

Now, by the statement S2(r) in Step III of the proof of Theorem 5.2, the sequence $\{\tilde{b}_i^{(r)}\}$ has to be bounded, so there must be some constant $K > 0$ such that the inequality below holds for every index i .

$$d_{i+1}^{(r)} \leq K d_i^{(r)} c_i^{(r)}$$

Hence $d_1^{(r)} \leq K d_0^{(r)} c_0^{(r)}$, $d_2^{(r)} \leq K(K d_0^{(r)} c_0^{(r)}) c_1^{(r)}$, $d_3^{(r)} \leq K(K^2 d_0^{(r)} c_0^{(r)} c_1^{(r)}) c_2^{(r)}$, and in general [for $i = i_r + j > i_r$, where i_r is as in the statement S4(r) in the proof of Theorem 5.2]:

$$d_i^{(r)} \leq d_0^{(r)} K^i \prod_{n=0}^{i-1} c_n^{(r)}$$

$$\begin{aligned}
&= d_0^{(r)} \left[K^{i_r} \prod_{n=0}^{i_r-1} c_n^{(r)} \right] \left[K^j \prod_{n=i_r}^{i_r+j-1} c_n^{(r)} \right] \\
&\leq d_0^{(r)} \left[K^{i_r} \prod_{n=0}^{i_r-1} c_n^{(r)} \right] K^j \varepsilon^{1+2+\dots+2^{j-1}} \\
&= d_0^{(r)} \left[K^{i_r} \prod_{n=0}^{i_r-1} c_n^{(r)} \right] \varepsilon^{-1} K^j \varepsilon^{2^j} (= \text{const} \cdot K^{i-i_r} \varepsilon^{2^{(i-i_r)}}).
\end{aligned}$$

It follows at once from the last inequality and from the estimate (63) that the sequence $\{res_{\overline{\Omega}}^{\Gamma}(\hat{\eta}^{(i)} - \hat{\eta}^{(0)})\}$ must be Cauchy within $\Gamma^r(\overline{\Omega}; L(s^*TM, T^\uparrow\Gamma))$ with respect to the C^r -norm topology. By Corollary A.5, the sequence $\{res_{\Omega}^{\Gamma}(\hat{\eta}^{(i)} - \hat{\eta}^{(0)})\}$ must then be Cauchy within $\Gamma^r(\Omega; L(s^*TM, T^\uparrow\Gamma))$ relative to the C^r -topology.

Since M can be covered with open subsets U of the kind considered in the preceding paragraph, by the same argument as in the proof of Theorem 5.2 it follows that the sequence $\{\hat{\eta}^{(i)} - \hat{\eta}^{(0)}\}$ must be Cauchy within $\Gamma^p(\Gamma; L(s^*TM, T^\uparrow\Gamma))$ (with respect to the C^p -topology) and hence that the sequence $\{\hat{\Psi}^{(i)}\}$ has to be convergent, within the affine Fréchet manifold $\text{Conn}^p(\Gamma)$, to a unique groupoid connection $\hat{\Psi}^{(\infty)}$ of class C^p . We contend that $\hat{\Psi}^{(\infty)}$ has to be multiplicative. Indeed, let $\hat{\eta}^{(\infty)}$ denote its horizontal lift. Since C^p -convergence implies pointwise convergence, we have $\hat{\eta}_{1x}^{(\infty)} = \lim_i \hat{\eta}_{1x}^{(i)} = \lim_i T_x 1 = T_x 1$ for every base point $x \in M$, because every $\hat{\Psi}^{(i)}$ is unital. Hence $\hat{\Psi}^{(\infty)}$ is unital. Notice that for each $g \in \Gamma$ we have $T_g t \circ \hat{\eta}_g^{(\infty)} = \lim_i T_g t \circ \hat{\eta}_g^{(i)} = \lim_i \hat{\lambda}_g^{(i)} = \hat{\lambda}_g^{(\infty)}$. Since for every divisible pair $(g, h) \in \Gamma_{\div}$ the ratio operation restricts to a linear (hence continuous) map $\div_{g,h} : T_g \Gamma \oplus_{T_{sg=sh}M} T_h \Gamma \rightarrow T_{gh^{-1}} \Gamma$, on the basis of (56) we compute:

$$\begin{aligned}
(\hat{\eta}_g^{(\infty)} \div \hat{\eta}_h^{(\infty)}) - \hat{\eta}_{gh^{-1}}^{(\infty)} \circ \hat{\lambda}_h^{(\infty)} &= \lim_i \{(\hat{\eta}_g^{(i)} \div \hat{\eta}_h^{(i)}) - \hat{\eta}_{gh^{-1}}^{(i)} \circ \hat{\lambda}_h^{(i)}\} \\
&= \lim_i \{\omega_{gh^{-1}}^{-1} \circ \Delta^{\hat{\Psi}^{(i)}}(g, h) \circ \hat{\lambda}_h^{(i)}\} \\
&= \omega_{gh^{-1}}^{-1} \circ \lim_i \{\Delta^{\hat{\Psi}^{(i)}}(g, h)\} \circ \hat{\lambda}_h^{(\infty)}.
\end{aligned}$$

Now $\lim_i \Delta^{\hat{\Psi}^{(i)}}(g, h) = 0$, because by the above estimates the sequence $\{d_i^{(0)}\}$ tends to zero. This shows that $\hat{\eta}^{(\infty)}$ satisfies the identity (14), and hence that $\hat{\Psi}^{(\infty)}$ (being unital, as we know already) is multiplicative. \square

Appendix A. Uniform convergence topologies on spaces of sections

Fix some order of differentiability $k \in \{0, 1, 2, \dots, \infty\}$; the value of this parameter is supposed to remain unchanged throughout the appendix.

Let E be a (real or complex) differentiable vector bundle over a differentiable manifold X . We shall say that a cross-section $\xi : S \rightarrow E$, defined over an arbitrary subset $S \subset X$, is of class C^k if for each point $x \in S$ the following is true:

- (*) There is some open neighborhood B of x in X to which $\xi|_{S \cap B}$ can be extended by a cross-section of E of class C^k .

We shall let $\Gamma^k(S; E)$ denote the vector space formed by all cross-sections of E over S of class C^k . Of course, when S is a submanifold of X (in particular, when S is an open

subset) the present notation is consistent with the notations introduced in Section 1; however notice that ‘class C^0 ’ does not agree with ‘continuous’ in general unless S is locally closed. For any subset T of S , we shall let res_T^S denote the linear map of $\Gamma^k(S; E)$ into $\Gamma^k(T; E)$ given by restriction from S to T . Observe that the map

$$\text{res}_S^{\bar{S}} : \Gamma^k(\bar{S}; E) \longrightarrow \Gamma^k(S; E)$$

(where \bar{S} denotes the closure of S in X) is injective and identifies $\Gamma^k(\bar{S}; E)$ with the linear subspace of $\Gamma^k(S; E)$ consisting of all those ξ such that the property (*) holds for each point x in X (not just for each point x in S).

I. C^k -Topology. We assume that the reader is familiar with some of the basic notions of the theory of topological vector spaces, such as for instance the notion of ‘locally convex topology generated by a family of seminorms’ or the notion of Fréchet space, which are discussed thoroughly in Chapter II of [23]. Acquaintance with the elementary concepts and examples of the theory of Fréchet manifolds, a good self-contained account of which can be found in Part I of [11], is advisable albeit not indispensable. Finally, for a description of the C^k -topology on the space of mappings between two differentiable manifolds, compare [19].

Suppose that $\varphi : U \xrightarrow{\cong} \varphi U \subset \mathbb{R}^n$ is a C^∞ local coordinate chart for a given differentiable manifold X . Also let E be a differentiable vector bundle over X and suppose that $\tau : E|U \xrightarrow{\cong} U \times \mathbb{K}^N$ is a C^∞ local vector bundle trivialization for E over the domain U of the chart φ . We can express an arbitrary global cross-section ξ of our vector bundle locally over U in terms of its components relative to τ .

$$\xi^\tau = (\xi_1^\tau, \dots, \xi_N^\tau) \stackrel{\text{def}}{=} pr_2 \circ \tau \circ \xi|U \quad \xi^{\tau, \varphi} = (\xi_1^{\tau, \varphi}, \dots, \xi_N^{\tau, \varphi}) \stackrel{\text{def}}{=} \xi^\tau \circ \varphi^{-1}$$

Obviously $\xi|U \in \Gamma^k(U; E)$ if and only if every component $\xi_I^{\tau, \varphi}$ is a function of class C^k defined on the open subset φU of \mathbb{R}^n and with values in \mathbb{K} . For every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ of order $|\alpha| = \alpha_1 + \dots + \alpha_n \leq k$ and for every function $f : \Omega \rightarrow \mathbb{K}$ of class C^k which is defined on some open domain $\Omega \subset \mathbb{R}^n$ we adopt the customary notational shorthand $\partial^\alpha f := \frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}} f$ (conventionally we set $\frac{\partial^0}{\partial t_i^0} f := f$). Let K be an arbitrary compact subset of U . For each natural number $r \leq k$ and for every global cross-section $\xi \in \Gamma^k(X; E)$ we set

$$p_r^{\tau, \varphi, K}(\xi) \stackrel{\text{def}}{=} \max_{0 \leq s \leq r} \max_{\alpha \in \mathbb{N}^n, |\alpha|=s} \sup_{x \in K} \max_{l=1, \dots, N} |\partial^\alpha \xi_l^{\tau, \varphi}(\varphi x)|.$$

Evidently this expression defines a seminorm $p_r^{\tau, \varphi, K}$ on the vector space $\Gamma^k(X; E)$. The topology of k -th order local uniform convergence—shortly, C^k -topology—on $\Gamma^k(X; E)$ is the locally convex vectorial topology generated by all those seminorms $p_r^{\tau, \varphi, K}$ that one obtains by letting τ, φ, K vary over all possible choices of a local vector bundle trivialization τ , a local coordinate chart φ , and a compact domain K as above, and by letting r vary over all natural numbers $\leq k$. Since E is locally trivial and X is locally compact, this topology is necessarily separated (i.e. Hausdorff).

Proposition A.1. *Suppose that X is a second countable differentiable manifold. Let E be an arbitrary differentiable vector bundle over X . The C^k -topology makes $\Gamma^k(X; E)$ into a Fréchet space (= complete, metrizable, and locally convex, topological vector space). When X is compact and k is finite, the Fréchet space $\Gamma^k(X; E)$ is actually Banachable (= complete normable). \square*

As we will do for various other statements in this appendix, we will leave the proof as an exercise. (One has to pay some attention here because the intersection of two compact subsets of a non-Hausdorff manifold need not itself be compact; for instance, consider the line with a double origin and therein the two compact intervals $[-1, 1]$ centered each around a different origin.)

II. Standard norms. In practice, it is convenient to work with a slightly more flexible definition of ‘generating seminorm for the C^k -topology’. By a *continuous norm* p on a differentiable vector bundle E over a differentiable manifold X we shall mean the datum, on each vector bundle fiber E_x , of a norm $p_x : E_x \rightarrow \mathbb{R}_{\geq 0}$ depending on x in such a manner that the function on E given by $e \mapsto p_{pr_x^E(e)}(e)$ is continuous.

Lemma A.2. *Let η_1, \dots, η_N be a local frame for E defined over some open set $U \subset X$. Let K be a compact subset of U . Let p be a continuous vector bundle norm on $E|U$. Then there exists some constant $c > 0$ such that if for any cross-section $\xi \in \Gamma^0(U; E)$ we write $\xi = \sum_I a_I \eta_I$ with $a_I \in C^0(U)$ then $\max_I \|a_I\|_K \leq c \sup_{x \in K} p_x(\xi(x))$.*

Proof. Put

$$c^{-1} = \inf_{x \in K} \inf_{|z_1|^2 + \dots + |z_N|^2 = 1} p_x(z_1 \eta_1(x) + \dots + z_N \eta_N(x)).$$

Whenever $|a_I(x)| > 0$ for some point $x \in K$ and for some index I , say $I = 1$, put $b_I = a_I(x)/a_1(x)$ for all I and let $\rho^2 = 1 + |b_2|^2 + \dots + |b_N|^2 \geq 1$. Then $(1/\rho)^2 + |b_2/\rho|^2 + \dots + |b_N/\rho|^2 = 1$, whence $c^{-1} \leq \rho c^{-1} \leq p_x(\eta_1(x) + b_2 \eta_2(x) + \dots + b_N \eta_N(x))$ and therefore $c^{-1} |a_1(x)| \leq p_x(\xi(x))$. \square

Examples. Any (Riemannian or Hermitian, depending on whether E is real or complex) metric ϕ on E gives rise to a continuous norm on E defined on each fiber E_x by the rule $e \mapsto \sqrt{\phi_x(e, e)}$. Next, any two continuous norms p and q on two \mathbb{K} -linear differentiable vector bundles E and F over X give rise, on the hom vector bundle $L(E, F)$, to a continuous norm defined on each fiber $L(E, F)_x = L(E_x, F_x)$ by the rule $\lambda \mapsto \sup_{p_x(e)=1} q_x(\lambda e)$. [The reader may find the following remarks useful: Let \mathbb{E} and \mathbb{F} be arbitrary vector spaces over \mathbb{K} . Let p_0 and p_1 be norms on \mathbb{E} such that $b p_0 \leq p_1 \leq B p_0$ for some constants $b, B > 0$, and let q_0 and q_1 be norms on \mathbb{F} such that $c q_0 \leq q_1 \leq C q_0$ for some other constants $c, C > 0$. The two norms on $L(\mathbb{E}, \mathbb{F})$ defined by the expression $\|\lambda\|_i = \sup_{p_i(e)=1} q_i(\lambda e)$ (as $i = 0, 1$) are related by the inequalities

$$c B^{-1} \|\cdot\|_0 \leq \|\cdot\|_1 \leq C b^{-1} \|\cdot\|_0.$$

In particular, if for some small positive real number ϵ we have $\min\{b, c\} \geq 1 - \epsilon$ and $\max\{B, C\} \leq 1 + \epsilon$, then $\frac{1-\epsilon}{1+\epsilon} \|\cdot\|_0 \leq \|\cdot\|_1 \leq \frac{1+\epsilon}{1-\epsilon} \|\cdot\|_0$.]

Let X, E be as above. By a *standard normed chart* for E we mean a triplet (φ, τ, p) consisting of: (i) a local coordinate chart φ of the form $U \xrightarrow{\sim} \mathbb{R}^n$ for X ; (ii) a local vector bundle trivialization $\tau : E|U \xrightarrow{\sim} U \times \mathbb{K}^N$ for E over the domain of definition of φ ; (iii) a continuous vector bundle norm $p : E|U \rightarrow \mathbb{R}_{\geq 0}$ on the restriction of E over U . For each local coordinate chart φ as in (i) and for each real number $a > 0$ we let B_a^φ and K_a^φ denote the two subsets of the chart domain $U \approx \mathbb{R}^n$ that correspond under φ to the two balls $B_a(0) \subset \overline{B}_a(0)$ (open, resp., closed) of radius a centered at the origin in \mathbb{R}^n . We also abbreviate ‘ B_1^φ ’ into ‘ B^φ ’ and ‘ K_1^φ ’ into ‘ K^φ ’. Suppose that Ω is a relatively compact open subset of X ; by definition, its closure $\overline{\Omega}$ is compact. By a *standard normed atlas* for E over $\overline{\Omega}$ we shall mean a finite collection $\mathcal{A} = \{(\varphi_i, \tau_i, p_i) \mid i \in \mathcal{I}\}$ of standard normed charts for E such that $\overline{\Omega} \subset \bigcup_{i \in \mathcal{I}} B^{\varphi_i}$.

Let X, E and Ω be as in the preceding paragraph. Suppose that we are given some standard normed atlas $\mathcal{A} = \{(\varphi_i, \tau_i, p_i) \mid i \in \mathcal{I}\}$ for E over $\overline{\Omega}$ with, say, coordinate charts $\varphi_i : U_i \xrightarrow{\sim} \mathbb{R}^{n_i}$ and vector bundle trivializations $\tau_i : E|U_i \xrightarrow{\sim} U_i \times \mathbb{K}^{N_i}$. For every index $i \in \mathcal{I}$ let us set $K_i = K^{\varphi_i}$, $B_i = B^{\varphi_i}$, and $\mathbb{E}_i = \mathbb{K}^{N_i}$. For each point $u \in U_i$ let $\| \cdot \|_{i,u}$ indicate the norm on \mathbb{E}_i corresponding to $p_{i,u}$ under the linear bijection $\tau_{i,u} : E_u \xrightarrow{\sim} \mathbb{E}_i$. Consider an arbitrary cross-section $\xi \in \Gamma^k(\overline{\Omega}; E)$. A fortiori, ξ will be defined over each intersection $\Omega_i = \Omega \cap U_i$ and hence will correspond via τ_i, φ_i to a unique \mathbb{E}_i -valued function ξ^{τ_i, φ_i} on $\varphi_i \Omega_i$ of class C^k . Since $\varphi_i \Omega_i$ is an open subset of euclidean space \mathbb{R}^{n_i} , for every multi-index $\alpha \in \mathbb{N}^{n_i}$ of order $|\alpha| \leq k$ the partial derivative $\partial^\alpha \xi^{\tau_i, \varphi_i}$ will make sense as an \mathbb{E}_i -valued function on $\varphi_i \Omega_i$. Then, for each natural number $r \leq k$, it will make sense to set

$$\|\xi\|_{C^r(\overline{\Omega}; E, \mathcal{A})} \stackrel{\text{def}}{=} \max_{i \in \mathcal{I}} \max_{\alpha \in \mathbb{N}^{n_i}, |\alpha| \leq r} \sup_{u \in B_i \cap \Omega} |\partial^\alpha \xi^{\tau_i, \varphi_i}(\varphi_i u)|_{i,u}. \quad (64)$$

We shall refer to the function $\|\cdot\|_{C^r(\overline{\Omega}; E, \mathcal{A})}$ thus defined on $\Gamma^k(\overline{\Omega}; E)$ as a *standard C^r -norm*. [Of course, in order for the expression (64) to define a norm, each one of the (finitely many) suprema occurring in (64) must be $< \infty$. This is the case because ξ is defined over all of $\overline{\Omega} \cap U_i$ (which is a closed subset of the second countable Hausdorff manifold $U_i \approx \mathbb{R}^{n_i}$) and, within U_i , is locally C^k -extendible around each point of that closed subset, so by the existence of partitions of unity it admits a C^k extension to all of U_i ($\supset K_i \supset B_i$).] Observe that our notation $\|\cdot\|_{C^r(\overline{\Omega}; E, \mathcal{A})}$ is correct in that the quantity (64) only depends on $\overline{\Omega}$, not on Ω ; the only relevant object, here, is a closed, compact subset of X which coincides with the closure of its own interior.

Proposition A.3. *Let E be an arbitrary differentiable vector bundle over a manifold X . Let Ω be a relatively compact open subset of X . For each natural number $r \leq k$, any two standard C^r -norms on $\Gamma^k(\overline{\Omega}; E)$ are equivalent.* \square

(Proof left as an exercise; you apply Lemma A.2 together with essentially the same arguments you use to prove Proposition A.1.) Let us temporarily refer to the locally convex vectorial topology on $\Gamma^k(\overline{\Omega}; E)$ generated by all standard C^r -norms as r varies over all natural numbers $\leq k$ as the ‘standard norm topology’. By a *C^r -norm* on $\Gamma^k(\overline{\Omega}; E)$ we shall mean a norm which is equivalent to some standard C^r -norm.

Corollary A.4. *Let X, E and Ω be as in the previous proposition. The restriction map $\text{res}_{\overline{\Omega}}^X : \Gamma^k(X; E) \rightarrow \Gamma^k(\overline{\Omega}; E)$ is continuous (relative to the C^k -topology on the first space and to the standard norm topology on the second space).* \square

Corollary A.5. *Let X , E and Ω be as in the previous proposition. The restriction map $\text{res}_{\Omega}^{\overline{\Omega}} : \Gamma^k(\overline{\Omega}; E) \rightarrow \Gamma^k(\Omega; E)$ is continuous (relative to the standard norm topology on the first space and to the C^k -topology on the second space). \square*

Observe that when X is compact these corollaries imply that the standard norm topology and the C^k -topology on $\Gamma^k(X; E)$ coincide. We shall henceforth refer to the standard norm topology on $\Gamma^k(\overline{\Omega}; E)$ as the C^k -topology, too.

III. Change of coefficients and change of variables.

Lemma A.6. *Let $\omega : E \rightarrow F$ be an arbitrary morphism between two \mathbb{K} -linear differentiable vector bundles over a given manifold X . The \mathbb{K} -linear map*

$$\Gamma^k(X; \omega) : \Gamma^k(X; E) \longrightarrow \Gamma^k(X; F), \quad \xi \mapsto \omega \circ \xi$$

is C^k -continuous, i.e., continuous relative to the C^k -topology on $\Gamma^k(X; E)$ and to the C^k -topology on $\Gamma^k(X; F)$. Furthermore, for any relatively compact open subset Ω of X , the linear map

$$\Gamma^k(\overline{\Omega}; \omega) : \Gamma^k(\overline{\Omega}; E) \longrightarrow \Gamma^k(\overline{\Omega}; F), \quad \xi \mapsto \omega \circ \xi$$

is C^k -continuous. \square

The two C^k -continuous linear maps in the lemma are compatible (by definition):

$$\text{res}_{\Omega}^X \circ \Gamma^k(X; \omega) = \Gamma^k(\overline{\Omega}; \omega) \circ \text{res}_{\Omega}^X. \quad (65)$$

By construction, for any differentiable mapping $f : Y \rightarrow X$ and for any differentiable vector bundle E over X , the pullback vector bundle f^*E has the fiber product $Y \times_X E$ as its total manifold and the first projection $Y \times_X E \rightarrow Y$ as its bundle projection onto Y . Clearly, a mapping $Z \rightarrow Y \times_X E$ is of class C^k if and only if its two components $Y \leftarrow Z \rightarrow E$ are both C^k . Hence, by the universal property of the fiber product, for each C^k cross-section ξ of E there will be a unique C^k cross-section $f^*\xi$ of f^*E such that $\text{pr}_E \circ f^*\xi = \xi \circ f$, where pr_E denotes the projection $Y \times_X E \rightarrow E$.

Lemma A.7. *Let $f : Y \rightarrow X$ be an arbitrary differentiable mapping. Also, let E be an arbitrary differentiable vector bundle over X . The cross-section pullback operation gives rise to a C^k -continuous linear map*

$$\Gamma^k(f; E) : \Gamma^k(X; E) \longrightarrow \Gamma^k(Y; f^*E), \quad \xi \mapsto f^*\xi.$$

Moreover, for any relatively compact open subset Ω of X and for any similar subset O of Y such that $f(O) \subset \Omega$, the linear map

$$\Gamma^k(f; O \rightarrow \Omega; E) : \Gamma^k(\overline{\Omega}; E) \longrightarrow \Gamma^k(\overline{O}; f^*E), \quad \xi \mapsto f^*\xi|_{\overline{O}}$$

is C^k -continuous. \square

The two C^k -continuous linear maps in the lemma are compatible (by definition):

$$\text{res}_{\Omega}^Y \circ \Gamma^k(f; E) = \Gamma^k(f; O \rightarrow \Omega; E) \circ \text{res}_{\Omega}^Y. \quad (66)$$

IV. Norm estimates for composition and inversion. To begin with, we introduce a notational device which will spare us the nuisance of keeping track of irrelevant scaling factors throughout. Let S be an arbitrary set. We introduce a binary relation \leq on the set $\text{Func}_{\geq 0}(S)$ of all non-negative real valued functions on S by defining $f \leq g$ to mean ‘there exists some constant $C > 0$ such that $f \leq Cg$ ’. Since this binary relation is reflexive and transitive, setting $f \equiv g \Leftrightarrow (f \leq g \ \& \ g \leq f)$ gives rise to an equivalence relation \equiv on $\text{Func}_{\geq 0}(S)$. Note that \leq descends to a partial order \leq on the set of all \equiv -equivalence classes of functions. Also note that for all $f, g, h \in \text{Func}_{\geq 0}(S)$

$$f \equiv g \text{ entails } f + h \equiv g + h \text{ and } fh \equiv gh.$$

Furthermore note that if $\lambda : S' \rightarrow S$ is any mapping then $f \equiv g \in \text{Func}_{\geq 0}(S)$ implies $f \circ \lambda \equiv g \circ \lambda \in \text{Func}_{\geq 0}(S')$. Thus the operations of sum, product and pullback make sense for \equiv -equivalence classes of functions. Now, for E, Ω and r as in the statement of Proposition A.3, let $\| \cdot \|_{C^r \overline{\Omega}; E}$ (or simply $\| \cdot \|_{C^r \overline{\Omega}}$ or even $\| \cdot \|_{C^r}$ when omission does not lead to confusion) denote the \equiv -class of any C^r -norm within $\text{Func}_{\geq 0}(\Gamma^k(\overline{\Omega}; E))$.

Lemma A.8. *Let E, F and G be \mathbb{K} -linear differentiable vector bundles over the same manifold X . Let Ω be a relatively compact open subset of X . Let η denote a variable ranging over $\Gamma^k(\overline{\Omega}; L(E, F))$ and ϑ one ranging over $\Gamma^k(\overline{\Omega}; L(F, G))$. The following estimates hold, in the second of which l is an arbitrary natural number $\leq k - 1$.**

$$\|\vartheta \circ \eta\|_{C^0} \leq \|\vartheta\|_{C^0} \|\eta\|_{C^0} \quad (67a)$$

$$\|\vartheta \circ \eta\|_{C^{l+1}} \leq \|\vartheta\|_{C^l} \|\eta\|_{C^{l+1}} + \|\vartheta\|_{C^{l+1}} \|\eta\|_{C^l} \quad (67b)$$

[*The intended meaning of these inequalities should be clear; by way of example, the first inequality is to be understood as follows:

$$\| \cdot \|_{C^0 \overline{\Omega}; L(E, G)} \circ \omega \leq (\| \cdot \|_{C^0 \overline{\Omega}; L(F, G)} \circ pr_1)(\| \cdot \|_{C^0 \overline{\Omega}; L(E, F)} \circ pr_2)$$

where pr_1 and pr_2 denote the two projections $\vartheta, \eta \mapsto \vartheta$ and $\mapsto \eta$, respectively, and where ω denotes the “composition” operation $\vartheta, \eta \mapsto \vartheta \circ \eta$.]

Proof. First of all, some preliminary considerations. Let $V \subset \mathbb{R}^n$ be an open subset of euclidean n -space and let $\mathbb{E}, \mathbb{F}, \mathbb{G}$ be vector spaces of finite dimension over \mathbb{K} . If $f : V \rightarrow L(\mathbb{E}, \mathbb{F})$ and $g : V \rightarrow L(\mathbb{F}, \mathbb{G})$ are two mappings of class C^1 defined on V then, letting $g \circ f$ (with a slight abuse of notation) denote the C^1 mapping of V into $L(\mathbb{E}, \mathbb{G})$ given by $y \mapsto g(y) \circ f(y)$, we have

$$\partial_j(g \circ f) = (\partial_j g) \circ f + g \circ (\partial_j f) \quad (68)$$

for every index $j = 1, \dots, n$. From (68), arguing by induction on the order $|\gamma|$ of a multi-index $\gamma \in \mathbb{N}^n$, we infer that if f, g are continuously differentiable up to order $|\gamma|$ then

$$\partial^\gamma(g \circ f) = \partial^\gamma g \circ f + \sum_{\gamma = \beta + \alpha, |\alpha| > 0} l_{\beta, \alpha} \partial^\beta g \circ \partial^\alpha f, \quad (69)$$

where the coefficients $l_{\beta, \alpha}$ are suitable non-negative integer constants.

Suppose (φ, τ, p) , (φ, σ, q) and (φ, ρ, r) are standard normed charts respectively for E , F and G defined over the same local coordinate patch $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$. Let us write $\tau : E|U \xrightarrow{\sim} U \times \mathbb{E}$, $\sigma : F|U \xrightarrow{\sim} U \times \mathbb{F}$, and $\rho : G|U \xrightarrow{\sim} U \times \mathbb{G}$. We have the induced local trivialization $[\tau, \sigma]$ for $L(E, F)$ over U given by

$$L(E, F)|U \cong L(E|U, F|U) \cong L(U \times \mathbb{E}, U \times \mathbb{F}) \cong U \times L(\mathbb{E}, \mathbb{F})$$

and the continuous norm $[p, q]$ on $L(E, F)|U$ given on each fiber $L(E_u, F_u)$ by

$$\lambda \mapsto \sup_{p_u(e)=1} q_u(\lambda e).$$

Similarly we have the induced standard normed chart $(\varphi, [\sigma, \rho], [q, r])$ for $L(F, G)$ and an analogous chart for $L(E, G)$. Next, suppose we are given two cross-sections $\eta \in \Gamma^k(\overline{\Omega}; L(E, F))$ and $\vartheta \in \Gamma^k(\overline{\Omega}; L(F, G))$. For every multi-index $\gamma \in \mathbb{N}^n$ of order $|\gamma| \leq l+1$ and for every point $u \in \Omega \cap U$ the identity (69) yields the inequality

$$\begin{aligned} \|\partial^\gamma(\vartheta \circ \eta)^{[\tau, \rho], \varphi}(\varphi u)\|_u &= \|\partial^\gamma(\vartheta^{[\sigma, \rho], \varphi} \circ \eta^{[\tau, \sigma], \varphi})(\varphi u)\|_u \\ &\leq \|\partial^\gamma \vartheta^{[\sigma, \rho], \varphi}(\varphi u)\|_u \|\eta^{[\tau, \sigma], \varphi}(\varphi u)\|_u \\ &\quad + \sum_{\gamma=\beta+\alpha, |\alpha|>0} l_{\beta, \alpha} \|\partial^\beta \vartheta^{[\sigma, \rho], \varphi}(\varphi u)\|_u \|\partial^\alpha \eta^{[\tau, \sigma], \varphi}(\varphi u)\|_u; \end{aligned} \quad (70)$$

each occurrence of $\|\cdot\|_u$, here, refers to the norm on $L(\mathbb{E}, \mathbb{F})$ [resp., $L(\mathbb{E}, \mathbb{G})$, $L(\mathbb{F}, \mathbb{G})$] corresponding to the norm $[p, q]_u$ (resp., $[p, r]_u$, $[q, r]_u$) under the linear bijection $[\tau, \sigma]_u$ (resp., $[\tau, \rho]_u$, $[\sigma, \rho]_u$).

To finish, let us choose any three standard normed atlases $\mathcal{A} = \{(\varphi_i, \tau_i, p_i) \mid i \in \mathcal{I}\}$, $\mathcal{B} = \{(\psi_j, \sigma_j, q_j) \mid j \in \mathcal{J}\}$ and $\mathcal{C} = \{(\chi_k, \rho_k, r_k) \mid k \in \mathcal{K}\}$ respectively for E , F and G over $\overline{\Omega}$. It is no loss of generality to assume that $\mathcal{I} = \mathcal{J} = \mathcal{K}$ and that $\varphi_i = \psi_i = \chi_i$ for all $i \in \mathcal{I}$. (To achieve this, simply select any finite family of standard coordinate patches for X which cover $\overline{\Omega}$ with their open unit balls, subordinate to the open cover of $\overline{\Omega}$ given by all triple intersections $U_i \cap V_j \cap W_k$, where U_i , V_j and W_k denote the domains of φ_i , ψ_j and χ_k , respectively.) Over $\overline{\Omega}$ there will be three induced standard normed atlases $[\mathcal{A}, \mathcal{B}] = \{(\varphi_i, [\tau_i, \sigma_i], [p_i, q_i]) \mid i \in \mathcal{I}\}$ for $L(E, F)$, $[\mathcal{B}, \mathcal{C}] = \dots$ for $L(F, G)$ and so forth. If we pick the standard norms associated to these standard normed atlases, we immediately deduce the desired estimates (67) from (70). \square

Lemma A.9. *Let E and F be two \mathbb{K} -linear differentiable vector bundles over a given manifold X . Let Ω be a relatively compact open subset of X . Let η denote a variable ranging over $\Gamma^k(\overline{\Omega}; \text{Lis}(E, F))$, where $\text{Lis}(E, F)$ denotes the open subset of $L(E, F)$ formed by all linear isomorphisms. Then, for each natural number $l \leq k-1$,*

$$\|\eta^{-1}\|_{C^{l+1}} \leq (\|\eta^{-1}\|_{C^1})^2 \|\eta\|_{C^{l+1}}. \quad (71)$$

Proof. Observe that if $f : V \rightarrow \text{Lis}(\mathbb{E}, \mathbb{F})$ is any mapping of class C^1 defined on some open subset $V \subset \mathbb{R}^n$ of euclidean n -space with values in the set of invertible linear maps between two finite-dimensional vector spaces \mathbb{E} and \mathbb{F} over \mathbb{K} then for every index $j = 1, \dots, n$, letting f^{-1} (by abuse of notation) denote the C^1 mapping of V into $\text{Lis}(\mathbb{F}, \mathbb{E})$ given by $y \mapsto f(y)^{-1}$, we have the identity

$$\partial_j(f^{-1}) = -f^{-1} \circ \partial_j f \circ f^{-1}. \quad (72)$$

For each multi-index $\gamma \in \mathbb{N}^n$ of positive order $|\gamma|$ not exceeding the order of differentiability of f , we deduce an identity of the following kind from (72) by making repeated use of the equation (68):

$$\partial^\gamma(f^{-1}) = - \sum_{\gamma=\beta+\alpha, |\alpha|>0} \sum_{\beta=\beta_1+\beta_2} l_{\beta_1, \beta_2}^\alpha \partial^{\beta_1}(f^{-1}) \circ \partial^\alpha f \circ \partial^{\beta_2}(f^{-1}) \quad (73)$$

(the coefficients $l_{\beta_1, \beta_2}^\alpha$ being appropriate non-negative integer constants).

Suppose (φ, τ, p) and (φ, σ, q) are standard normed charts respectively for E and F defined over the same coordinate patch $\varphi : U \xrightarrow{\sim} \mathbb{R}^n$. In the notations of the preceding proof, for each multi-index $\gamma \in \mathbb{N}^n$ of order $|\gamma| > 0, \leq l + 1$ the equation (73) implies the following inequality, which is supposed to be valid for every point $u \in \Omega \cap U$ for any given cross-section $\eta \in \Gamma^k(\bar{\Omega}; \text{Lis}(E, F))$:

$$\begin{aligned} \|\partial^\gamma(\eta^{-1})^{[\sigma, \tau], \varphi}(\varphi u)\|_u &= \|\partial^\gamma[(\eta^{[\tau, \sigma], \varphi})^{-1}](\varphi u)\|_u \\ &\leq \sum_{\gamma=\beta+\alpha, |\alpha|>0} \sum_{\beta=\beta_1+\beta_2} l_{\beta_1, \beta_2}^\alpha \|\partial^{\beta_1}[(\eta^{[\tau, \sigma], \varphi})^{-1}](\varphi u)\|_u \|\partial^\alpha \eta^{[\tau, \sigma], \varphi}(\varphi u)\|_u \\ &\quad \times \|\partial^{\beta_2}[(\eta^{[\tau, \sigma], \varphi})^{-1}](\varphi u)\|_u \\ &= \sum_{\gamma=\beta+\alpha, |\alpha|>0} \sum_{\beta=\beta_1+\beta_2} l_{\beta_1, \beta_2}^\alpha \|\partial^{\beta_1}(\eta^{-1})^{[\sigma, \tau], \varphi}(\varphi u)\|_u \\ &\quad \times \|\partial^{\beta_2}(\eta^{-1})^{[\sigma, \tau], \varphi}(\varphi u)\|_u \|\partial^\alpha \eta^{[\tau, \sigma], \varphi}(\varphi u)\|_u. \end{aligned}$$

[The case $|\gamma| = 0$ must be handled separately:

$$\begin{aligned} \|(\eta^{-1})^{[\sigma, \tau], \varphi}(\varphi u)\|_u &= \|(\eta^{-1})^{[\sigma, \tau], \varphi}(\varphi u) \circ \eta^{[\tau, \sigma], \varphi}(\varphi u) \circ \eta^{[\tau, \sigma], \varphi}(\varphi u)^{-1}\|_u \\ &\leq \|(\eta^{-1})^{[\sigma, \tau], \varphi}(\varphi u)\|_u^2 \|\eta^{[\tau, \sigma], \varphi}(\varphi u)\|_u. \end{aligned}$$

From this point on, our argument substantially follows the same pattern as in the proof of Lemma A.8. \square

Appendix B. Haar integrals depending on parameters

Haar systems enable one to construct invariant “integration functionals” on Lie groupoids. In spite of their name, Haar systems bear a much closer relation to the original nineteenth-century geometric construction of invariant integrals on Lie groups in terms of invariant volume densities rather than to Haar’s more abstract, purely topological construction. Even though this aspect is well known to experts, it is only rarely made explicit in the literature [21, 20, 28, 3], most of which concerns topological groupoids. For this reason, and also because we do not know any reference where the fundamental lemma about Haar integrals depending on parameters (namely, Proposition B.12 and B.13 below) is stated in the form needed for the purposes of the present work, we are going to provide a self-contained account of Haar integration.

As in the preceding appendix, let us once and for all fix some order of differentiability $k \in \{0, 1, 2, \dots, \infty\}$.

I. Integration of densities along the fibers of a submersion. Let E be an arbitrary real differentiable vector bundle over a differentiable manifold X . For any real number $s > 0$, the s -density bundle on E , hereafter denoted by $\Delta^s E$, is the differentiable real line bundle over X constructed as follows. For each point $x \in X$, let $r(x) = \text{rk}_E(x)$ indicate the rank of E at x . Define $\Delta^s E_x$ to be the (real) vector space formed by all the functions $h : \bigwedge^{r(x)} E_x \rightarrow \mathbb{R}$ such that $h(tw) = |t|^s h(w)$ for every $t \in \mathbb{R}$ and for every $w \in \bigwedge^{r(x)} E_x$. Set $\Delta^s E := \coprod_{x \in X} \Delta^s E_x$ (disjoint union). By definition, a C^∞ local trivialization for $\Delta^s E$ over the domain of definition U of an arbitrary C^∞ local trivializing frame $\xi = (\xi_1, \dots, \xi_r)$ for E is provided by the mapping

$$\coprod_{u \in U} \Delta^s E_u \ni (u, h) \mapsto (u, \langle h, \xi_1(u) \wedge \dots \wedge \xi_r(u) \rangle) \in U \times \mathbb{R}.$$

By an s -density of class C^k on E we shall mean an arbitrary global cross-section of $\Delta^s E$ of class C^k .

Suppose we are given a C^∞ mapping $\phi : E' \rightarrow E$ between two real differentiable vector bundles E' and E which sends fibers into fibers, thus inducing a differentiable mapping say $f : X' \rightarrow X$ of the base X' of E' into the base X of E , and which for each point $x' \in X'$ sets up a linear bijection $\phi_{x'} : E'_{x'} \xrightarrow{\sim} E_{f(x')}$ between the fiber of E' at x' and the corresponding fiber of E . Of course we may interpret ϕ as a vector bundle isomorphism between E' and f^*E (the pullback of E along f), although this point of view is less convenient when coming to notations. For any given s -density $\delta \in \Gamma^k(X; \Delta^s E)$ of class C^k on E , we have a new s -density on E' , which we shall call the *inverse image* of δ under ϕ and indicate by $\phi^*\delta$, itself of class C^k , given by

$$(\phi^*\delta)(x') := \delta(f(x')) \circ \bigwedge^{r(x')} \phi_{x'}.$$

In the special case when ϕ is the canonical projection from a pullback $f^*E \rightarrow E$, we shall normally write $f^*\delta$ for the inverse image of δ under ϕ , and refer to this as the *pullback* of δ along f .

If γ is a Riemannian metric of class C^∞ on a real differentiable vector bundle E , there exists on E a unique 1-density δ of class C^∞ with the property that $\delta(x)(e_1 \wedge \dots \wedge e_r) = 1$ for every base point x of E and for every γ_x -orthonormal vector space basis $\{e_1, \dots, e_r\} \subset E_x$. We shall refer to δ as the *volume density* associated to the metric γ and adopt the notation Vol_γ for it. Note that for $\phi : E' \rightarrow E$ as in the preceding paragraph we have $\phi^*\text{Vol}_\gamma = \text{Vol}_{\phi^*\gamma}$ (where $\phi^*\gamma$ denotes the inverse image metric). In general, we shall call ‘volume density on E ’ any 1-density δ of class C^∞ on E having the property that $\delta(x)w > 0$ for all x and for all non-zero $w \in \bigwedge^{r(x)} E_x$. In the sequel, we shall only deal with volume densities. It will be convenient to simply write ‘ ΔE ’ in place of ‘ $\Delta^1 E$ ’, and to abbreviate ‘ ΔTX ’ into ‘ ΔX ’ for any differentiable manifold X . We shall also be speaking about volume densities “on X ”, really meaning “on the tangent bundle of X ”.

Let $\phi : Z \rightarrow Y$ be a surjective submersion between two Hausdorff manifolds. Let $T^{\downarrow\phi}Z$ denote the ϕ -vertical tangent bundle of Z , that is, the differentiable subbundle of TZ given (as z varies over Z) by $T_z^{\downarrow\phi}Z := \ker(T_z\phi : T_zZ \rightarrow T_{\phi z}Y)$. By a *volume density along the fibers of ϕ* , or a ϕ -vertical volume density, we shall mean a volume density on the ϕ -vertical tangent bundle of Z . We shall say that a subset S of Z is

ϕ -properly located if the restriction $\phi|_S : S \rightarrow Y$ is a proper mapping (the inverse images of compact sets are compact sets). We shall say that a function f on Z is ϕ -properly supported if its support $\text{supp}_Z f$ is ϕ -properly located. Of course, when ϕ is fixed throughout a discussion, we may abbreviate ' $T^\downarrow \phi Z$ ' into ' $T^\downarrow Z$ ' and refer to this simply as the “vertical tangent bundle” of Z ; similarly for the rest of the terminology introduced in the present paragraph.

There is a pairing, called *integration along the fiber*, between continuous functions with properly located support and vertical volume densities. Let $f \in C(Z)$ be any such function, and let $\delta \in \Gamma^\infty(Z; \Delta T^\downarrow Z)$ be any such density. For each point y in Y , there is a canonical isomorphism of differentiable line bundles over $\phi^{-1}(y)$

$$(\Delta T^\downarrow Z)|_{\phi^{-1}(y)} \cong \Delta \phi^{-1}(y)$$

induced at each point $z \in \phi^{-1}(y)$ by the vector space identification $T_z \phi^{-1}(y) = T_z^\downarrow Z$. The restriction of δ to $\phi^{-1}(y)$ can be regarded as an ordinary volume density on $\phi^{-1}(y)$, which for brevity we shall indicate by δ_y . Now, letting μ_y denote the positive Radon measure on $\phi^{-1}(y)$ attached to δ_y in the standard way (i.e. through the localization theorem for positive functionals and the change of variables formula, as in [15, p. 451]), we obtain a function $\int f \delta$ on Y upon integrating f against μ_y for variable y :

$$y \mapsto \int_{\phi^{-1}(y)} f|_{\phi^{-1}(y)} d\mu_y.$$

Lemma B.1. *Let Y and Z be Hausdorff differentiable manifolds, and let $\phi : Z \rightarrow Y$ be a surjective and submersive differentiable mapping. Let $\delta \in \Gamma^\infty(Z; \Delta T^\downarrow Z)$ be a volume density along the fibers of ϕ . Then:*

- (a) *If f is a properly supported C^k function on Z , the function $\int f \delta$ on Y is also of class C^k .*
- (b) *For any properly located subset S of Z , the operation of integration along the fiber gives rise to a C^k -continuous linear map $\int(-)\delta : C_S^k(Z) \rightarrow C^k(Y)$, where $C_S^k(Z)$ denotes the closed linear subspace of $C^k(Z)$ formed by all those functions f such that $\text{supp}_Z f \subset S$.*

Proof. To begin with, let us prove the lemma in the special case when $Y = U$ is an open subset of \mathbb{R}^n , ϕ is the projection from a product $\mathbb{R}^m \times U = Z$ on the 2nd factor $U \subset \mathbb{R}^n$, and S is a subset of Z of the form $K \times U$ with K a compact set in \mathbb{R}^m . Let $x = (x_1, \dots, x_m)$ denote the coordinates in \mathbb{R}^m and $y = (y_1, \dots, y_n)$ those in \mathbb{R}^n . In the situation just described, our density δ can be written

$$\delta(x, y) = \rho(x, y) dx = \rho(x, y) dx_1 \cdots dx_m$$

where ρ is some positive function of class C^∞ . By definition, for all $y \in U$ we have $(\int f \delta)(y) = \int_{\mathbb{R}^m} f(x, y) \rho(x, y) dx$ (integration with respect to Lebesgue measure), $f \in C^k(\mathbb{R}^m \times U)$ being an arbitrary properly supported function of class C^k . Differentiation under the integral sign shows that $\int f \delta$ belongs to $C^k(U)$, thus establishing our first

claim (a). By the same token, for any compact set $L \subset U$, natural number $r \leq k$, and multi-index $\alpha \in \mathbb{N}^n$ of order $|\alpha| \leq r$, and for every function $f \in C_S^k(\mathbb{R}^m \times U)$, we have

$$\begin{aligned} \sup_{y \in L} \left| \partial^\alpha \left(\int f \delta \right)(y) \right| &= \sup_{y \in L} \left| \int_{\mathbb{R}^m} \partial^{(0,\alpha)}(f\rho)(x, y) dx \right| \\ &\leq \text{meas}(K) \cdot \sup_{(x,y) \in K \times L} |\partial^{(0,\alpha)}(f\rho)(x, y)| \\ &\leq \text{meas}(K) \cdot p_r^{K \times L}(f\rho). \end{aligned}$$

Since by Lemma A.6 the linear map of $C_S^k(\mathbb{R}^m \times U)$ into itself given by $f \mapsto f\rho$ is C^k -continuous, our second claim (b) is also established for the special case under exam.

Let us now consider the general case. Suppose we are given a local coordinate chart $\psi : V \xrightarrow{\sim} \mathbb{R}^n$ for Y . The closed “ball” $A = \psi^{-1}(\overline{B}_1(0))$ is a compact subset of Y . It follows that $\phi^{-1}(A) \cap S$ is a compact subset of Z , because S is properly located. We can therefore cover $\phi^{-1}(A) \cap S$ with a finite family of local trivializing charts for ϕ , say,

$$\begin{array}{ccc} Z \supset \text{open } W_i & \xrightarrow{\sim} & \mathbb{R}^{m_i} \times \mathbb{R}^n \\ \downarrow \phi|_{W_i} & & \downarrow p_r \\ V \supset \text{open } \tilde{V}_i & \xrightarrow{\sim} & \mathbb{R}^n \end{array} \quad [i \in I \text{ finite set}].$$

Let us choose some C^∞ partition of unity $\{g_j\}$ on $W = \bigcup_{i \in I} W_i$, subordinated to $\{W_i\}$, with compact supports $\text{supp } g_j \subset W_{i(j)}$. Then, let us select some finite set J of j -indices so that $\sum_{j \in J} g_j = 1$ in a neighborhood of $\phi^{-1}(A) \cap S$. Next, for each $i \in I$, let us put $g_i = \sum_{\{j \in J | i(j)=i\}} g_j \in C_c^\infty(W_i)$. Of course, we must have $\sum_{i \in I} g_i = 1$ in a neighborhood of $\phi^{-1}(A) \cap S$. Let us identify g_i notationally with its extension by zero to all of Z . We contend that for each $i \in I$ the correspondence $f \mapsto \int g_i f \delta$ determines a C^k -continuous linear map of $C^k(Z)$ into $C^k(Y)$. This is a straightforward consequence of the special case considered above and of the two lemmas A.6 and A.7, since the map in question can be decomposed in the following fashion

$$C^k(Z) \xrightarrow{(g_i \cdot)|_{W_i}} C_{\text{supp } g_i}^k(W_i) \xrightarrow{\int (-) \delta|_{W_i}} C_{\phi(\text{supp } g_i)}^k(V_i) \xrightarrow{\text{ext}} C^k(Y)$$

where ‘ext’ stands for ‘extension by zero’ (obviously a C^k -continuous linear map). Now, since for every function $f \in C_S^k(Z)$ we have $\int f \delta = \int \sum_i g_i f \delta$ near A , for any given natural number $r \leq k$ we conclude from the inequality

$$p_r^{\psi, A} \left(\int f \delta \right) = p_r^{\psi, A} \left(\int \sum_i g_i f \delta \right) \leq \sum_i p_r^{\psi, A} \left(\int g_i f \delta \right)$$

and from the C^k -continuity of every linear map $f \mapsto \int g_i f \delta$ that the linear map $f \mapsto \int f \delta$ restricted to $C_S^k(Z)$ is continuous with respect to the seminorm $p_r^{\psi, A}$ on $C^k(Y)$. \square

We shall need the following variant of integration along the fiber. Let $\phi : Z \twoheadrightarrow Y$ and $\delta \in \Gamma^\infty(Z; \Delta T^\downarrow Z)$ be as in the statement of the previous lemma, and let $g \in C^\infty(Z)$ be a properly supported differentiable function on Z . Let $W \Subset Z$ and $V \Subset Y$ be two relatively compact, open, subsets such that

$$\phi(W) \subset V \quad \text{and} \quad \phi^{-1}(V) \cap \text{supp } g \subset W. \quad (74)$$

Clearly, it is possible to find a relatively compact, open, neighborhood W_1 of \overline{W} such that $\phi^{-1}(\overline{V}) \cap \text{supp } g \subset W_1$. Let a function $f \in C^k(\overline{W})$ be given. By a standard partition of unity argument, f must admit some extension f_1 of class C^k over W_1 . For each point $y \in \overline{V}$, there must be some compact neighborhood B of y such that $\phi^{-1}(B) \cap \text{supp } g \subset W_1$, because $\phi^{-1}(y) \cap \text{supp } g \subset W_1$ and g is properly supported. Thus, there exists some relatively compact, open, neighborhood V_1 of \overline{V} with the property that the function $f_1 g$ (on W_1) extends by zero to a properly supported function of class C^k defined on the whole open tube $\phi^{-1}(V_1)$. By Lemma B.1, the latter function can be integrated against δ into a function $\int f_1 g \delta \in C^k(V_1)$, whose restriction to \overline{V} will depend only on f and not on the specific extension f_1 which we have used. We therefore get a well-defined element $\int f g \delta$ of $C^k(\overline{V})$.

Lemma B.2. *The linear map $\int(-)g\delta : C^k(\overline{W}) \rightarrow C^k(\overline{V})$ which we have just defined is C^k -continuous.*

Proof. The special case when ϕ is the projection $\mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g \in C_c^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ and we integrate against euclidean volume $dx_1 \cdots dx_m$ is dealt with as in the proof of the preceding lemma. Back to the general case, we observe that for any open subset $B \subset Z$ and compactly supported function $h \in C_c^\infty(Z)$ with $\text{supp } h \subset B$, setting $A = \phi(B)$,

$$gh \in C_c^\infty(Z), \quad W \cap B \Subset Z \quad \text{and} \quad V \cap A \Subset Y$$

satisfy the hypotheses (74); indeed,

$$\begin{aligned} \phi^{-1}(V \cap A) \cap \text{supp } gh &\subset \phi^{-1}(V) \cap \text{supp } g \cap \text{supp } h \\ &\subset W \cap \text{supp } h \subset W \cap B. \end{aligned}$$

Now, the “truncated” integration map $\int(-)gh\delta : C^k(\overline{W}) \rightarrow C^k(\overline{V})$ factors as

$$C^k(\overline{W}) \xrightarrow{\text{res}} C^k(\overline{W \cap B}) \xrightarrow{\int(-)gh\delta} C_{V \cap \phi(\text{supp } h)}^k(\overline{V \cap A}) \xrightarrow{\text{ext}} C^k(\overline{V}).$$

[The ‘extension by zero’ step makes sense because $\overline{V} \cap A$ is an open neighborhood of $\overline{V} \cap \phi(\text{supp } h)$ in \overline{V} .] The lemma then follows from the special case mentioned at the beginning, by the same partition of unity argument as in the proof of Lemma B.1. \square

II. Haar systems on differentiable groupoids.

Definition B.3. Let $\Gamma \rightrightarrows X$ be a Hausdorff differentiable groupoid. A *left Haar system* $\mu = \{\mu_x\}_{x \in X}$ on $\Gamma \rightrightarrows X$ assigns to each base point $x \in X$ a positive Radon measure μ_x on the target fiber $\Gamma_x = t^{-1}(x)$ in such a way that the following three conditions are met:

- (a) (*Differentiability.*) One can exhibit a volume density along the target fibers, say, τ with the property that for each base point x the measure μ_x on Γ_x coincides with the positive Radon measure associated to the volume density $\tau_x := \tau|_{t^{-1}(x)}$.
- (b) (*Left invariance.*) For any arrow $g \in \Gamma$, and for any Borel subset A of the target fiber Γ_{sg} , setting $gA = \{gh \mid h \in A\}$ one has

$$\mu_{tg}(gA) = \mu_{sg}(A).$$

(c) (*Definiteness.*) $\mu_x(U) > 0$ for every non-empty open subset U of Γ_x .

Comments. On a differentiable group, a left Haar system is the same thing as left Haar measure [15, p. 351]. The condition (c) is of course redundant, since it is implied by (a). There is an alternative formulation of the left invariance condition, which is often more useful in practice; for an arbitrary function f on Γ_{sg} , let $L_g f$ denote the left translate of f by g , that is, the function on Γ_{tg} defined by $(L_g f)(h) = f(g^{-1}h)$:

(b') For any arrow $g \in \Gamma$, the left translate $L_g f$ of an arbitrary μ_{sg} -integrable function $f \in \mathcal{L}^1(\mu_{sg})$ is an element of $\mathcal{L}^1(\mu_{tg})$, and

$$\int_{\Gamma_{tg}} L_g f d\mu_{tg} = \int_{\Gamma_{sg}} f d\mu_{sg}.$$

Proposition B.4. *Let $\Gamma \rightrightarrows X$ be a Hausdorff differentiable groupoid which is based on a second countable manifold X . Then $\Gamma \rightrightarrows X$ admits a left Haar system.*

Proof. It will be more convenient for us to construct a *right* Haar system. (Any right Haar system can be turned into a left Haar system by means of the groupoid inversion mapping. We leave the straightforward details to the reader.) As in Section 2, let $\mathfrak{g} = 1^* T^\uparrow \Gamma$ denote the algebroid bundle of Γ . Since by assumption X is a second countable Hausdorff manifold, it admits C^∞ partitions of unity. As a consequence, the (real) differentiable vector bundle \mathfrak{g} over X can be endowed with a C^∞ Riemannian metric, say, γ . Letting $\omega : T^\uparrow \Gamma \xrightarrow{\sim} \mathfrak{g}$ denote the Maurer–Cartan isomorphism associated to Γ , the inverse image $\omega^* \text{Vol}_{\mathfrak{g}, \gamma}$ will be a volume density along the source fibers, whose corresponding system of positive Radon measures will constitute a right Haar system on $\Gamma \rightrightarrows X$. \square

III. Normalizing functions.

Definition B.5. Suppose $\Gamma \rightrightarrows X$ is a Hausdorff differentiable groupoid. Let $\mu = \{\mu_x\}$ be a left Haar system on $\Gamma \rightrightarrows X$. A *normalizing* (or *cut-off*) *function* for μ is a C^∞ non-negative function $\kappa : X \rightarrow [0, +\infty)$ with the following two properties:

(a) The composite function $\kappa \circ s : \Gamma \rightarrow [0, +\infty)$ is t -properly supported.

(b) $\int \kappa \circ s d\mu = 1$ (= constant function of value one on X).

We shall call the pair $\nu = (\mu, \kappa)$ a *normalized left Haar system* on Γ .

Let $\nu = (\mu, \kappa)$ be a normalized left Haar system on a Hausdorff differentiable groupoid $\Gamma \rightrightarrows X$. For each point x in X the function $[\kappa \circ s]_x = (\kappa \circ s)|_{\Gamma_x}$ on the target fiber Γ_x belongs to $C_c(\Gamma_x) \subset \mathcal{L}^1(\mu_x)$. Consider the finite positive Radon measure ν_x on Γ_x that by Riesz' theorem corresponds to the bounded positive functional on $C_c(\Gamma_x)$ given by $g \mapsto \int g[\kappa \circ s]_x d\mu_x$. A Borel-measurable function f on Γ_x lies in $\mathcal{L}^1(\nu_x)$ if and only if the product function $f[\kappa \circ s]_x$ lies in $\mathcal{L}^1(\mu_x)$, in which case $\int f d\nu_x = \int f[\kappa \circ s]_x d\mu_x$. Thus $C(\Gamma_x) \subset \mathcal{L}^1(\nu_x)$. By the same token, the system of measures $\{\nu_x\}$ is left invariant. By abuse of language and of notation, we shall refer to $\{\nu_x\}$ too as a “normalized left Haar system” and write $\nu = \{\nu_x\}$.

Example. Let $\Gamma \rightrightarrows X$ be any Hausdorff differentiable groupoid whose target mapping is proper (the inverse image under t of any compact subset of X is compact; this is the case e.g. when the manifold Γ is compact). Let $\mu = \{\mu_x\}$ be any left Haar system on $\Gamma \rightrightarrows X$. The function κ on X defined by $\kappa(x) = 1 / \int 1 d\mu_x$ is normalizing for μ . Note that $\kappa \circ s = \kappa \circ t$, by the left invariance of μ . The normalized system $\nu = \{\nu_x\}$ is itself a left Haar system. Furthermore, it is a *probability* system: $\nu_x(\Gamma_x) = 1$ for all x .

A differentiable groupoid $\Gamma \rightrightarrows X$ is *proper* if it is Hausdorff and its anchor $(s, t) : \Gamma \rightarrow X \times X$, $g \mapsto (sg, tg)$ is a proper mapping.

Proposition B.6. *Let $\Gamma \rightrightarrows X$ be a proper differentiable groupoid over a second countable manifold X . Any Haar system on $\Gamma \rightrightarrows X$ admits normalizing functions.*

Proof. Let X/Γ denote the orbit space of Γ . As a set, this is the quotient of X by the equivalence relation $x \equiv y \Leftrightarrow \exists g \in \Gamma (x = sg \text{ \& } y = tg)$. Its topology is the finest making the quotient projection $\pi : X \rightarrow X/\Gamma$ continuous. It is evident that π is an open mapping and hence that X/Γ is a locally compact space. The properness of Γ implies that X/Γ is Hausdorff. The second countability of X implies that X/Γ is second countable.

Fix any sequence of compact sets $A_0 = \emptyset, A_1, A_2, \dots$ in X/Γ such that $A_i \subset \text{Int}(A_{i+1})$ and such that $X/\Gamma = \bigcup_{i=1}^{\infty} A_i$. (Compare [15, p. 341, proof of Theorem 11].) For each $i \geq 1$, set $U_i = \pi^{-1}(\text{Int}(A_{i+2}) \setminus A_{i-1})$. Since the manifold X is second countable and Hausdorff, it is possible to find a C^∞ partition of unity $\{V_j, \psi_j\}$ with compact supports subordinated to the open cover $\{U_i\}$. Put $W_j = \{x \in X \mid \psi_j(x) > 0\}$. For each non-negative integer i , it is possible to find a finite set, say, $J(i)$ of j -indices so that the corresponding open sets $\pi(W_j)$ cover the compact set $A_{i+1} \setminus \text{Int}(A_i)$ and each of them intersects it non-vacuously. Set $J = \bigcup_{i=0}^{\infty} J(i)$. Clearly $\{\pi(W_j)\}_{j \in J}$ must be a locally finite open cover of X/Γ . Each point x in X must possess an invariant open neighborhood intersecting only finitely many open sets V_j with $j \in J$. If we set $\chi = \sum_{j \in J} \psi_j$ then the function $\chi \circ s$ must be t -properly supported and non-zero along every t -fiber.

Let $\mu = \{\mu_x\}$ be a left Haar system on $\Gamma \rightrightarrows X$. We have $\int \chi \circ s d\mu > 0$ (everywhere on X). It is evident that the following function is then normalizing for μ :

$$\kappa = \left(1 / \int \chi \circ s d\mu\right) \chi. \quad \square$$

IV. Adapted domains. For any differentiable groupoid $\Gamma \rightrightarrows X$, and for any pair S, T of subsets of its base X , we shall let $\Gamma(S, T)$ or Γ_T^S denote the subset of Γ consisting of all arrows g such that $sg \in S$ & $tg \in T$. We shall write $\Gamma|S \rightrightarrows S$ for the groupoid over S whose set of arrows is $\Gamma(S, S)$. We shall let ΓS denote the *saturation* of S , that is, the subset of X consisting of all points which are connected to points in S by arrows of $\Gamma \rightrightarrows X$.

Let us begin by highlighting the following trivial fact, already implicit in the proof of the preceding proposition.

Lemma B.7. *Let $\Gamma \rightrightarrows X$ be a proper differentiable groupoid. The following assertions are equivalent, κ being an arbitrary continuous function on X :*

- (a) *The function $\kappa \circ s$ on Γ is t -properly supported.*

(b) The set $\text{supp } \kappa \cap \Gamma K$ is compact for every compact $K \subset X$.

Proof. Since s is an open mapping, $\text{supp}(\kappa \circ s) = s^{-1}(\text{supp } \kappa)$. The identity

$$\text{supp } \kappa \cap \Gamma K = s(s^{-1}(\text{supp } \kappa) \cap t^{-1}(K))$$

then yields the implication (a) \Rightarrow (b). The converse implication (b) \Rightarrow (a) follows from the properness of Γ , because

$$\begin{aligned} \text{supp}(\kappa \circ s) \cap t^{-1}(K) &= (s, t)^{-1}(\text{supp } \kappa \times K) \\ &= (s, t)^{-1}([\text{supp } \kappa \cap \Gamma K] \times K). \end{aligned} \quad \square$$

Terminology. We shall say that an open subset U of the base X of a differentiable groupoid $\Gamma \rightrightarrows X$ is *adapted* to a continuous function κ on X , or that it is an *adapted domain* for κ , if $\text{supp } \kappa \cap \Gamma U \subset U$.

Observe that if U is adapted to κ then, setting $\Omega = \Gamma | U \rightrightarrows U$, for every $S \subset U$

$$\text{supp}_U(\kappa | U) \cap \Omega S = \text{supp } \kappa \cap \Gamma S. \quad (75)$$

Lemma B.8. *Let κ be a continuous function on the base X of a proper differentiable groupoid $\Gamma \rightrightarrows X$. Let $U \subset X$ be an open subset which is adapted to κ . Set $\Omega = \Gamma | U \rightrightarrows U$, $U' = \Gamma U$, and $\Omega' = \Gamma | U' \rightrightarrows U'$. The following properties are equivalent:*

- (a) *The function $(\kappa | U) \circ s^\Omega$ on Ω is t^Ω -properly supported.*
- (b) *The function $(\kappa | U') \circ s^{\Omega'}$ on Ω' is $t^{\Omega'}$ -properly supported.*

Proof. Use the identity (75) and Lemma B.7. \square

Lemma B.9. *Let $\Gamma \rightrightarrows X$ be a proper differentiable groupoid and let U be an open subset of X such that $\Gamma U = X$. Put $\Omega = \Gamma | U \rightrightarrows U$. Let κ be a C^∞ function on U such that the function $\kappa \circ s^\Omega \in C^\infty(\Omega)$ is t^Ω -properly supported. Then, there is a unique function $\tilde{\kappa} \in C^\infty(X)$ such that $\text{supp } \tilde{\kappa} \subset U$ and $\tilde{\kappa} | U = \kappa$.*

Proof. Using Lemma B.7, it is not hard to see that $\text{clos}_X\{u \in U \mid \kappa(u) \neq 0\} \subset U$. It follows that the extension of κ by zero to all of X is a function of class C^∞ . \square

Notation. Let $\Gamma \rightrightarrows X$ be a Hausdorff differentiable groupoid, and let μ be a left Haar system on $\Gamma \rightrightarrows X$. For any open subset U of X , there is on the groupoid $\Omega = \Gamma | U \rightrightarrows U$ an induced left Haar system $\mu | U$ defined at each point u in U by setting $(\mu | U)_u(A) = \mu_u(A)$ for every Borel subset A of the target fiber $\Omega_u = \Gamma_u \cap s^{-1}(U)$.

Proposition B.10. *Let $\Gamma \rightrightarrows X$ be a proper differentiable groupoid, and let μ be a left Haar system on $\Gamma \rightrightarrows X$. Let $U \subset X$ be an open subset such that $\Gamma U = X$. Suppose that $\kappa \in C^\infty(U, \mathbb{R}_{\geq 0})$ is a normalizing function for $\mu | U$. Then the extension of κ by zero, which by Lemma B.9 is a function $\tilde{\kappa}$ in $C^\infty(X, \mathbb{R}_{\geq 0})$, is a normalizing function for μ .*

Proof. Since obviously U is adapted to $\tilde{\kappa}$, Lemma B.8 shows that the function $\tilde{\kappa} \circ s$ is t -properly supported. Since s is an open mapping, $\text{supp}(\tilde{\kappa} \circ s) = s^{-1}(\text{supp } \tilde{\kappa}) \subset s^{-1}(U)$. Given a point x in X , let us choose an arrow g in $\Gamma(x, U)$ ($\neq \emptyset$ since $\Gamma U = X$) and then put $u = tg$. By the left invariance of μ , setting $\Omega = \Gamma|U \rightrightarrows U$, we have

$$\begin{aligned} \int_{\Gamma_x} (\tilde{\kappa} \circ s)_x d\mu_x &= \int_{\Gamma_u} L_g(\tilde{\kappa} \circ s)_x d\mu_u \\ &= \int_{\Gamma_u} (\tilde{\kappa} \circ s)_u d\mu_u \\ &= \int_{\Omega_u} (\tilde{\kappa} \circ s)_u d\mu_u = \int_{\Omega_u} (\kappa \circ s^\Omega)_u d(\mu|U)_u = 1. \end{aligned} \quad \square$$

Proposition B.11. *Let $\Gamma \rightrightarrows X$ be a proper differentiable groupoid endowed with a normalized left Haar system $\nu = (\mu, \kappa)$. Let $U \subset X$ be an open subset which is adapted to the normalizing function κ . Then $\nu|U := (\mu|U, \kappa|U)$ is a normalized left Haar system on the groupoid $\Gamma|U \rightrightarrows U$.*

Proof. In the notations of Lemma B.8, the function $(\kappa|U) \circ s^\Omega$ is t^Ω -properly supported. Read in reverse order, the computation in the proof of the previous proposition shows that $\kappa|U$ is a normalizing function for $\mu|U$. \square

Terminology. For arbitrary normalized left Haar systems $\nu = (\mu, \kappa)$ we shall be talking about ‘ ν -adapted open sets’ or ‘adapted domains for ν ’. Of course, in all such phrases ‘ κ ’ should be understood in place of ‘ ν ’.

V. Dependence on global parameters under the integral sign.

Proposition B.12. *Let $\Gamma \rightrightarrows X$ be a differentiable groupoid which is Hausdorff and second countable and on which a normalized left Haar system $\nu = (\mu, \kappa)$ is assigned. Let $f : P \rightarrow X$ be an arbitrary differentiable mapping from some “space of parameters” P into the base manifold X of the groupoid. Let E be an arbitrary differentiable vector bundle over P . Then, letting pr_P denote the projection from the fiber product $P \times_{f \times \iota} \Gamma = \{(y, h) \in P \times \Gamma \mid f(y) = th\}$ on the first factor P , any global C^k cross-section ϑ of the vector bundle pullback pr_P^*E (over the manifold $P \times_{f \times \iota} \Gamma$) can be turned into a global C^k cross-section $\int \vartheta d\nu$ of the vector bundle E by integration along the target fibers:*

$$P \ni y \mapsto \left(\int \vartheta d\nu \right)(y) \stackrel{\text{def}}{=} \int_{th=f(y)} \vartheta(y, h) d\nu_{f(y)}(h) \in E_y \quad (76a)$$

(the integrand being a vector-valued continuous function on the target fiber $\Gamma_{f(y)}$ with values in the finite-dimensional vector space E_y). The resulting “integration functional”

$$d_{f;E}^k \nu : \Gamma^k(P \times_{f \times \iota} \Gamma; pr_P^*E) \longrightarrow \Gamma^k(P; E), \quad \vartheta \mapsto \langle \vartheta, d_{f;E}^k \nu \rangle := \int \vartheta d\nu \quad (76b)$$

is a C^k -continuous linear map.

Proof. Let η_1, \dots, η_N be a local trivializing frame (of C^∞ cross-sections) for E defined over the domain V of some local coordinate chart $\psi : V \xrightarrow{\cong} \psi V \subset \mathbb{R}^p$ for P , and let $\tilde{\eta} : E|V \xrightarrow{\cong} V \times \mathbb{K}^N$ denote the corresponding vector bundle trivialization. The pullback cross-sections $pr^*\eta_1, \dots, pr^*\eta_N$ (henceforth, for brevity, $pr = pr_P$) constitute a local trivializing frame for pr^*E over the open subset $W = pr^{-1}(V) = V \times_{f \times \iota} \Gamma$ of $P \times_{f \times \iota} \Gamma$;

let $pr^*\tilde{\eta}$ denote the corresponding vector bundle trivialization. For every cross-section $\vartheta \in \Gamma(P_f \times_t \Gamma; pr^*E)$ let us write

$$\vartheta \mid W = \sum_{I=1}^N \vartheta_I^{pr^*\tilde{\eta}} pr^*\eta_I$$

(the functions $\vartheta_I^{pr^*\tilde{\eta}} : W \rightarrow \mathbb{K}$ being uniquely determined). Evidently, $\vartheta \mid W$ is of class C^k precisely when $\vartheta_I^{pr^*\tilde{\eta}} \in C^k(W, \mathbb{K})$ for every $I = 1, \dots, N$. By definition of the notion of ‘integration of a vector-valued function’, for all $y \in V$

$$(\int \vartheta d\nu)(y) = \sum_{I=1}^N \left[\int_{th=f(y)} \vartheta_I^{pr^*\tilde{\eta}}(y, h) d\nu_{f(y)}(h) \right] \eta_I(y)$$

and, therefore,

$$\left(\int \vartheta d\nu \right)_I^{\tilde{\eta}} = \int_{th=f(-)} \vartheta_I^{pr^*\tilde{\eta}}(-, h) d\nu_{f(-)}(h) = \int \vartheta_I^{pr^*\tilde{\eta}} d\nu \quad (I = 1, \dots, N).$$

Hence, for any compact set $K \subset V$, and for every integer $r \in \mathbb{N}_{\leq k}$,

$$p_r^{\tilde{\eta}, \psi, K} \left(\int \vartheta d\nu \right) = \max_{I=1, \dots, N} p_r^{\psi, K} \left(\left(\int \vartheta d\nu \right)_I^{\tilde{\eta}} \right) = \max_{I=1, \dots, N} p_r^{\psi, K} \left(\int \vartheta_I^{pr^*\tilde{\eta}} d\nu \right).$$

We are thus reduced to the case when $E = \underline{\mathbb{K}}_P$ is a trivial line bundle over P (“trivial coefficients”). We may—and shall—assume that we are dealing with functions, rather than with cross-sections of an arbitrary vector bundle.

Let $\tau \in \Gamma^\infty(\Gamma; \Delta^\perp \Gamma)$ be a volume density along the target fibers whose associated system of positive Radon measures is μ . Observe that the projection $pr_P : P_f \times_t \Gamma \rightarrow P$ is a surjective submersion. (It is the pullback of a surjective submersion.) Also notice that the tangent map $T(pr_\Gamma) : T(P_f \times_t \Gamma) \rightarrow T\Gamma$ (associated to the other projection $pr_\Gamma : P_f \times_t \Gamma \rightarrow \Gamma$) induces an isomorphism of vector bundles $T^{\downarrow pr_P}(P_f \times_t \Gamma) \cong pr_\Gamma^*(T^\perp \Gamma)$, by means of which the pullback $pr_\Gamma^*\tau$ can be regarded as a volume density along the fibers of pr_P . Now, by construction, in the case of trivial coefficients the integration functional (76b) can be written as the composition of two linear maps which are already known to be continuous (by Lemma A.6 and Lemma B.1, respectively):

$$C^k(P_f \times_t \Gamma) \xrightarrow{(\kappa \circ s \circ pr_\Gamma)^-} C_{\text{supp}(\kappa \circ s \circ pr_\Gamma)}^k(P_f \times_t \Gamma) \xrightarrow{\int (-) pr_\Gamma^* \tau} C^k(P). \quad \square$$

Proposition B.13. *Let $\Gamma \rightrightarrows X$ be a second countable, proper, differentiable groupoid, and let $\nu = (\mu, \kappa)$ be a normalized left Haar system on $\Gamma \rightrightarrows X$. Let $f : P \rightarrow X$ be a differentiable mapping whose domain is a second countable, Hausdorff, manifold, and let E be a differentiable vector bundle over P . Suppose that U is a relatively compact, ν -adapted, non-empty, open subset of X and that V is a relatively compact, non-empty, open subset of P such that $f(V) \subset U$. Put $\Omega = \Gamma(U, U)$. Then, there exists a unique linear map $d_{f:V \rightarrow U; E}^k \nu$ from $\Gamma^k(\overline{V_f \times_t \Omega}; pr_P^*E)$ into $\Gamma^k(\overline{V}; E)$ which makes the diagram*

$$\begin{array}{ccc} \Gamma^k(P_f \times_t \Gamma; pr_P^*E) & \xrightarrow{d_{f: E}^k \nu} & \Gamma^k(P; E) \\ \downarrow \text{res} & & \downarrow \text{res} \\ \Gamma^k(\overline{V_f \times_t \Omega}; pr_P^*E) & \xrightarrow{d_{f: V \rightarrow U; E}^k \nu} & \Gamma^k(\overline{V}; E) \end{array} \quad (77)$$

commute. This linear map is C^k -continuous.

Proof. Since all manifolds involved in the statement are assumed to be Hausdorff and second countable, a straightforward partition of unity argument shows that the restriction maps in (77) are surjective. There will therefore be at most one solution to the problem represented by (77). To see that one such solution exists, we have to check that for every cross-section $\vartheta \in \Gamma^k(P_{f \times_t \Gamma}; pr_P^* E)$ which vanishes on $V_{f \times_t \Omega}$ the cross-section $\int \vartheta d\nu \in \Gamma^k(P; E)$ vanishes on V . Now for each $y \in f^{-1}(U)$ we have $\kappa(sh) = 0$ for all $h \in \Gamma_{f(y)} \setminus s^{-1}(U)$, because U is adapted to κ . So, when $y \in V$,

$$\begin{aligned} \int_{th=f(y)} \vartheta(y, h) d\nu_{f(y)}(h) &= \int_{th=f(y)} \vartheta(y, h) \kappa(sh) d\mu_{f(y)}(h) \\ &= \int_{h \in \Gamma_{f(y)} \cap s^{-1}(U)} \vartheta(y, h) \kappa(sh) d\mu_{f(y)}(h) = 0. \end{aligned}$$

We proceed to show the C^k -continuity of the linear map $d_{f:V \rightarrow U; E}^k \nu$. Reduction to the case of trivial coefficients is clear when V is so small that its closure \bar{V} lies within the domain of definition of some local trivializing frame for the vector bundle E . For a general open set V , we first choose a finite cover $\{V_i\}$ of \bar{V} by relatively compact open sets so that E trivializes around each closure \bar{V}_i , and then a partition of unity $\{g_i\}$ over \bar{V} subordinated to this cover in the sense that $\text{supp } g_i \subset V_i$ and $\sum_i g_i = 1$ on \bar{V} . Now

$$\int (-) d\nu = \sum_i g_i \int (-) d\nu = \sum_i \int (g_i \circ pr_P) - d\nu,$$

where each linear map $\int (g_i \circ pr_P) - d\nu$ factors as

$$\begin{aligned} \Gamma^k(\overline{V_{f \times_t \Omega}}; pr_P^* E) &\xrightarrow{res} \Gamma^k(\overline{[V \cap V_i]_{f \times_t \Omega}}; pr_P^* E) \xrightarrow{\int (g_i \circ pr_P) - d\nu} \\ &\Gamma_{\bar{V} \cap \text{supp } g_i}^k(\overline{V \cap V_i}; E) \xrightarrow{ext} \Gamma^k(\bar{V}; E), \end{aligned}$$

the first map being restriction, the last extension by zero.

In the case of trivial coefficients, the linear map defined by the commutativity of (77) is of the type considered in Lemma B.2, namely, in the notations of that lemma, it arises upon taking: (i) ϕ to be the projection $pr_P : P_{f \times_t \Gamma} \rightarrow P$; (ii) δ to be the density $pr_P^* \tau$, with τ as in the proof of the preceding proposition (last paragraph); (iii) g to be the function $\kappa \circ s \circ pr_P$; (iv) W to be the relatively compact open set $V_{f \times_t \Omega} \Subset P_{f \times_t \Gamma}$. We only need to make sure that the function $\kappa \circ s \circ pr_P$ is pr_P -properly supported and that the hypotheses (74) are satisfied. To this end, observe that

$$\text{supp}(\kappa \circ s \circ pr_P) \subset pr_P^{-1}(\text{supp } \kappa \circ s),$$

whence for every subset $A \subset P$

$$\begin{aligned} pr_P^{-1}(A) \cap \text{supp}(\kappa \circ s \circ pr_P) &\subset pr_P^{-1}(A) \cap pr_P^{-1}(\text{supp } \kappa \circ s) \\ &= A_{f \times_t \Gamma} \cap \text{supp}(\kappa \circ s). \end{aligned} \tag{78}$$

If A is compact then $t^{-1}(f(A)) \cap \text{supp}(\kappa \circ s)$ is also compact since $\kappa \circ s$ is t -properly supported (by definition of normalizing function). This shows that $\kappa \circ s \circ pr_I$ is pr_P -properly supported. If, on the other hand, we take $A = V$ in (78) then, since $f(V) \subset U$ and since U is κ -adapted, we have (cf. the proof of Lemma B.7)

$$\text{supp}(\kappa \circ s) \cap t^{-1}(f(V)) \subset s^{-1}(\text{supp } \kappa) \cap t^{-1}(U) \subset \Omega. \quad \square$$

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